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The Feynman–Kac Representation and Dobrushin–Lanford–Ruelle States of a Quantum Bose-Gas

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Abstract: This paper focuses on infinite-volume bosonic states for a quantum particle system (a quantum gas) in $\mathbb{R}^d$. The kinetic energy part of the Hamiltonian is the standard Laplacian (with a boundary condition at the border of a 'box'). The particles interact with each other through a two-body finite-range potential depending on the distance between them and featuring a hard core of diameter $a > 0$. We introduce a class of so-called FK-DLR functionals containing all limiting Gibbs states of the system. As a justification of this concept, we prove that for $d = 2$, any FK-DLR functional is shift-invariant, regardless of whether it is unique or not. This yields a quantum analog of results previously achieved by Richthammer.

Keywords: bosonic quantum system; Hamiltonian; Laplacian; two-body interaction; finite-range potential; hard core; Fock space; FK-representation; density matrix; Gibbs state; reduced density matrix; thermodynamic limit; FK-DLR equations

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1. Introduction. Infinite-Volume Gibbs States and Reduced Density Matrices

The results of the paper and related works. In this paper we focus on bosonic quantum systems and pursue two directions of study: (i) a working definition of an infinite-volume quantum Gibbs state for various types of quantum bosonic systems, and (ii) its justification, which we have chosen to be the shift-invariance property for a 2D Bose-gas. The starting point for this work was the famous definition by Dobrushin–Lanford–Ruelle (DLR) of an infinite-volume Gibbs probability distribution, which is universally accepted in contemporary Mathematical Physics and beyond.

(i) Our approach combines the DLR-equation and the Ginibre representation of density matrix kernel [1] and develops the approach outlined in [2]. Alternative approaches are represented in [3–5], based on probability measures on distributions or direct analysis of the spectrum of the Hamiltonian. See also the biblio quoted in the above sources. The first result of the paper in this direction is the proof of existence, by compactness, of a compatible family of infinite-volume reduced density matrices for a given family of local Hamiltonians (1.1.1), under a natural super-stability-type condition (1.1.14) (cf. Theorem 1 in Section 1.3). Next, we establish that the kernels of the infinite-volume reduced density matrices satisfy an analog of the DLR equation, which we call an FK-DLR equation. (FK stands for Feynman–Kac.) Solutions to the FK-DLR equations can be considered as quantum analogs of infinite-volume Gibbs probability distributions (cf. Theorem 6 in Section 2.4).

(ii) The first exact facts about the absence of shift symmetry-breaking in 2D systems with short-range interactions appeared in [6]. A remarkable progress was achieved in [7–10], demonstrating that the thermal equilibrium states of 2D classical systems exhibit the shift-symmetries of their Hamiltonians. Our results in this direction are Theorems 2 and 3 in Section 1.3 extending the above shift-invariance properties to the quantum systems under consideration. We have been influenced by Refs. [11–13] providing a number of technical tools and insights used in the current text.

We would like to note the book [14]: although it focuses upon a different class of systems (quantum anharmonic oscillators as opposite to quantum gases), it gives a useful discussion of a number of tools and notions which are of a universal character, see also [15].

Main notation. (a) $H_{n,\Lambda}$: the local Hamiltonian of an n-particle system in the volume Λ; (b) $G_{n,\Lambda} = e^{-\beta H_{n,\Lambda}}$: the Gibbs operator; (c) $\Xi_{\beta.n}(\Lambda)$ and $\Xi_{\mathbf{z},\beta}(\Lambda) = \sum\limits_{n\geq 0} \mathrm{z}^n \Xi_{\beta,n}(\Lambda)$: the n-particle and the grand-canonical partition functions; (d) $\varphi_{\mathbf{z},\beta,\Lambda}$: the Gibbs state with the density matrix $R_{\mathbf{z},\beta,\Lambda}$; (e) $\overline{\mathcal{W}}^*(x,y) = \bigcup\limits_{k\geq 1} \overline{\mathcal{W}}^{k\beta}(x,y)$: the space of continuous paths from x to y with the Wiener-bridge measure $\overline{\mathbb{P}}^*_{x,y}$; (f) $F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}$: the reduced density matrix kernel in $\Lambda_0 \subset \Lambda$ with the boundary condition $\mathbf{x}(\Lambda^c)$; (g) $F^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0)$: an infinite-volume density matrix kernel.

Organization of the paper. Sections 1.1 and 1.2 introduce the local Hamiltonian,quantum Gibbs state and its termodynamical limit. In Section 1.3 we present the main results, Theorems 1–3. In Section 2.1 we discuss the Feynman–Kac (FK)-representation for the reduced density matrix kernels (RDMKs) $F^{\Lambda_0}_{\Lambda}(\mathbf{x}_0, \mathbf{y}_0)$ and $F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}(\mathbf{x}_0, \mathbf{y}_0)$; in Section 2.2 we give the FK-representation for their infinite-volume counterparts $F^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0)$. On the basis of these representations, we define the class of FK-DLR states (more generally, FK-DLR functionals) and state Theorems 6 and 7, extending the assertions of Theorems 4 and 5 to this class. Sections 3 and 4 contain the outline of proofs. More technical elements of the proofs are presented in Sections 5–7. Finally, a bird's eye view of the subject and a direction of future research are discussed in Section 8.

1.1. The Local Hamiltonian

The object of this study is a quantum Bose-gas in a Euclidean space $\mathbb{R}^d$, $d \geq 2$. The starting point of our analysis is a self-adjoint n-particle Hamiltonian, $H_{n,\Lambda}$, in a finite 'box' Λ represented by a cube $[-\mathrm{L},\mathrm{L}]^d$, of size $2\mathrm{L} > 0$, centered at the origin. (Other types of bounded domains in $\mathbb{R}^d$ can/will also be incorporated.) Operator $H_{n,\Lambda}$ acts on functions $\underline{x}^n_1 = \{x(1),\ldots,x(n)\} \in \Lambda^n \mapsto \phi_n(\underline{x}^n_1)$ from $\mathrm{L}^{\mathrm{sym},a}_2(\Lambda^n)$ by

$$\left(H_{n,\Lambda}\phi_n\right)\left(\underline{x}^n_1\right) = -\frac{1}{2}\sum_{1\leq j\leq n}\left(\Delta_j\phi_n\right)\left(\underline{x}^n_1\right) + \sum_{1\leq j<j'\leq n} V\left(|x(j)-x(j')|\right)\phi_n\left(\underline{x}^n_1\right). \tag{1.1.1}$$

Here, $\mathrm{L}^{\mathrm{sym},a}_2(\Lambda^n)$ is the subspace in the Hilbert space $\mathrm{L}_2(\Lambda^n) = \mathrm{L}_2(\Lambda)^{\otimes n}$ formed by symmetric functions of variables $x(j)$, $1 \leq j \leq n$, constituting the argument $\underline{x}^n_1$, which vanish whenever

$$\min\ \left[\left|x(j)-x(j')\right|_{\mathrm{Eu}} :\ 1\leq j<j'\leq n\right] < a.$$

($|x|_{\mathrm{Eu}}$, or briefly $|x|$, stands for the Euclidean norm of $x \in \mathbb{R}^d$ whereas $|x|_{\mathrm{m}}$ denotes the max-norm.) Parameter $a > 0$ is fixed and represents the diameter of the hard core (see below). It is convenient to denote

$$\begin{aligned}\Lambda^n_a = \Big\{&\underline{x}^n_1 = \{x(1),\ldots,x(n)\} \in \Lambda^n :\\ &\min\ \left[|x(j)-x(j')| :\ 1\leq j<j'\leq n\right] \geq a\Big\}\end{aligned} \tag{1.1.2}$$

and identify $\mathrm{L}_2^{\mathrm{sym},a}(\Lambda^n)$ with $\mathrm{L}_2^{\mathrm{sym}}(\Lambda_a^n)$, the Hilbert space of square-integrable symmetric functions $\phi_n\left(\underline{x}_1^n\right)$ with support in Λ_a^n.

Operator Δ_j in (1.1.1) acts as a Laplacian in the variable $x(j)$. Further, $V : r \in [a,+\infty) \mapsto V(r) \in \mathbb{R}$ is a C^2-function describing a two-body interaction potential depending upon the distance between particles. Pictorially, we set: $V(r) = +\infty$ for $0 \le r < a$, conforming with the hard-core assumption. In the following condition (1.1.3a), we attempt to control the negative (attracting) part of $V(r)$: we assume that

$$\begin{aligned} &-\min[0,V(r)] \le W_-(r) \text{ where } r\in(0,\infty)\to W_-(r)\ge 0\\ &\text{is a decreasing function with } \mathrm{W}^- := \sum_{x\in\mathbb{Z}^d} W_-(a|x|) < \infty. \end{aligned} \tag{1.1.3a}$$

(A sufficient condition for $\mathrm{W}^- < \infty$ is that $\int\limits_0^\infty r^{d-1}W_-(r)\mathrm{d}r < \infty$.) Observe that when $V(r) \ge 0$ then $\mathrm{W}^- = 0$ (this includes the case of pure hard cores where $V(r) \equiv 0$ for $r \ge a$). In a similar manner, we assure a control over the derivative V':

$$\begin{aligned} &|V'(r)| \le W^{(1)}(r) \text{ where } r\in(a,\infty)\to W^{(1)}(r)\ge 0\\ &\text{is a decreasing function with } \mathrm{W}^{(1)} := \sum_{x\in\mathbb{Z}^d} W^{(1)}(a|x|) < \infty. \end{aligned} \tag{1.1.3b}$$

(A sufficient condition for $\mathrm{W}^{(1)<\infty}$ is that $\int\limits_0^\infty r^{d-1}W^{(1)}(r)\mathrm{d}r < \infty$.)Physically, one can say that the potential $V(r)$, $r > a$, has a bounded derivative and decays to 0 for large r in a qualified manner.

We also set

$$\overline{V}^+ = \max\left[V(r):\ r\ge a\right],\ \ -\overline{V}^- = \min\left[V(r):\ r\ge a\right], \tag{1.1.4a}$$

with $\overline{V}^- = 0$ for $V \ge 0$, and

$$\overline{V}^{(1)} = \max\left[\left|V'(r)\right| : r\ge a\right],\ \ \overline{V}^{(2)} = \max\left[\left|V''(r)\right| : r\ge a\right]. \tag{1.1.4b}$$

In the case where $V(r) = 0$ for $r \ge \mathrm{R}^\circ$, we say that V has a finite range; the smallest value $\mathrm{R}^\circ \in (0,\infty)$ with this property is called the interaction radius (or the interaction range) and is referred to in the relevant bounds.

For $n = 1$, the sum $\sum\limits_{1\le j<j'\le n}$ in Equation (1.1.1) is suppressed, and $H_{n,\Lambda}$ is reduced to $-\Delta/2$ in Λ. For $n = 0$, we formally set $H_{0,\Lambda} = 0$. In general, the term $-\frac{1}{2}\sum\limits_{1\le j\le n}\left(\Delta_j\phi_n\right)\left(\underline{x}_1^n\right)$ represents the kinetic energy part in the Hamiltonian, and the term $\sum\limits_{1\le j<j'\le n} V\left(|x(j)-x(j')|\right)$ the potential energy (as an operator, it is given as multiplication by this function). Note that if n is large enough (when n disjoint balls of diameter a can't be placed in a box Λ) then the expression for $H_{n,\Lambda}$ formally becomes infinite; so we will only care about the values of n such that the set $\Lambda_a^n \ne \emptyset$.

To complete the definition of operator $H_{n,\Lambda}$, we need to specify a boundary condition. More precisely, $H_{n,\Lambda}$ is initially defined by the right hand side (RHS) of Equation (1.1.1) as a symmetric operator on the set of C^2-functions $\phi = \phi_n$ with the support in the interior of Λ_a^n, see [16]. A self-adjoint extension of this symmetric operator emerges when we impose the Dirichlet boundary condition:

$$\phi(\underline{x}_1^n) = 0 \ \text{ for } \ \underline{x}_1^n = \{x(1),\dots,x(n)\} \in \partial^{(a)}\Lambda_a^n \cup \partial^{\mathrm{out}}\Lambda_a^n. \tag{1.1.5}$$

Here

$$\begin{aligned} \partial^{(a)}\Lambda_a^n &= \left\{\underline{x}_1^n \in \Lambda_a^n : \min\ \left[|x(j)-x(j')| :\ 1\le j<j'\le n\right] = a\right\},\\ \partial^{\mathrm{out}}\Lambda_a^n &= \left\{\underline{x}_1^n \in \Lambda_a^n : \max\ \left[|x(j)|_{\mathrm{m}} = L :\ 1\le j\le n\right] = a\right\}. \end{aligned} \tag{1.1.6}$$

Other examples of boundary conditions on $\partial^{\rm out}\Lambda_a^n$ for which the methods of this paper are applicable are Neumann and periodic. (In fact, one can incorporate general elastic boundary conditions. We intend to analyze these in a forthcoming work.)

In the Krein–Vishik classification, [17], Dirichlet's boundary condition generates a 'soft' self-adjoint extension whereas Neumann's boundary condition generates a 'rigid' self-adjoint extension. These two self-adjoint extensions are extreme ones (among Dirichlet-form extensions) in the sense of a natural order of the eigenvalues. Moreover, in our scheme the choice of the boundary condition for $H_{n,\Lambda}$ may vary from one square Λ to another (and even from one value of n to another). This endeavors towards inclusion of a broad class of Hamiltonians, aiming at enhancing possible phase transitions.

Under the above assumptions, operator $H_{n,\Lambda}$ is self-adjoint, bounded from below and has a pure point spectrum. Moreover, $\forall\ \beta\in(0,+\infty)$, the Gibbs operator $G_{\beta,n,\Lambda}=\exp\,[-\beta H_{n,\Lambda}]$ is a positive-definite trace-class operator in $\mathrm{L}_2^{\rm sym}(\Lambda_a^n)$. The trace

$$\Xi_{\beta,n}(\Lambda):=\mathrm{tr}_{\mathrm{L}_2^{\rm sym}(\Lambda_a^n)}G_{\beta,n,\Lambda}\in(0,+\infty) \tag{1.1.7}$$

is called the n-particle partition function in Λ at the inverse temperature β. When n is large and Λ_a^n becomes empty, we set $G_{\beta,n,\Lambda}$ to be a zero operator with $\Xi_{\beta,n}(\Lambda)=0$. This allows us to work with the grand canonical Gibbs ensemble. Namely, $\forall\ \mathrm{z}\in(0,+\infty)$, the direct sum

$$G_{\mathbf{z},\beta,\Lambda}=\bigoplus_{n\geq 0}\mathrm{z}^nG_{\beta,n,\Lambda} \tag{1.1.8}$$

determines a positive-definite trace-class operator in the bosonic Fock space

$$\mathcal{H}(\Lambda)=\bigoplus_{n\geq 0}\mathrm{L}_2^{\rm sym}(\Lambda_a^n). \tag{1.1.9}$$

The quantity

$$\Xi_{\mathbf{z},\beta}(\Lambda):=\sum_{n\geq 0}\mathrm{z}^n\Xi_{\beta,n}(\Lambda)=\mathrm{tr}_{\mathcal{H}(\Lambda)}G_{\mathbf{z},\beta,\Lambda}\in(0,+\infty) \tag{1.1.10}$$

is called the grand canonical partition function in Λ at fugacity z and the inverse temperature β. Further, the operator

$$R_{\beta,\Lambda}=\frac{1}{\Xi_{\mathbf{z},\beta}(\Lambda)}G_{\mathbf{z},\beta,\Lambda} \tag{1.1.11}$$

is called the (grand-canonical) density matrix (DM) in Λ; this is a positive-definite operator in $\mathcal{H}(\Lambda)$ of trace 1. Operator $R_{\mathbf{z},\beta,\Lambda}$ determines the Gibbs state (GS), i.e., a linear positive normalized functional $\varphi_{\mathbf{z},\beta,\Lambda}$ on the C*-algebra $\mathfrak{B}(\Lambda)$ of bounded operators in $\mathcal{H}(\Lambda)$ (see [2]):

$$\varphi_{\mathbf{z},\beta,\Lambda}(A)=\mathrm{tr}_{\mathcal{H}(\Lambda)}\left(AR_{\mathbf{z},\beta,\Lambda}\right),\quad A\in\mathfrak{B}(\Lambda). \tag{1.1.12a}$$

The next object of interest is the reduced DM (in short, the RDM), in volume $\Lambda_0\subset\Lambda$. We use this term for the partial trace

$$R^{\Lambda_0}_{\mathbf{z},\beta,\Lambda}=\mathrm{tr}_{\mathcal{H}(\Lambda\setminus\Lambda_0)}R_{\mathbf{z},\beta,\Lambda}; \tag{1.1.12b}$$

it is based on the tensor-product representation $\mathcal{H}(\Lambda)=\mathcal{H}(\Lambda_0)\otimes\mathcal{H}(\Lambda\setminus\Lambda_0)$. Operator $R^{\Lambda_0}_{\mathbf{z},\beta,\Lambda}$ acts in $\mathcal{H}(\Lambda_0)$, is positive-definite and has trace 1. Moreover, the partial trace operation leads to an important compatibility property for RDMs: if $\Lambda_1\subset\Lambda_0\subset\Lambda$ then

$$R^{\Lambda_1}_{\mathbf{z},\beta,\Lambda}=\mathrm{tr}_{\mathcal{H}(\Lambda_0\setminus\Lambda_1)}R^{\Lambda_0}_{\mathbf{z},\beta,\Lambda}. \tag{1.1.13}$$

The main results of the present paper are valid for any given $\mathrm{z},\beta\in(0,\infty)$ under the condition

$$\rho:=\mathrm{z}\exp\left(\beta\mathrm{W}^-\right)<1, \tag{1.1.14}$$

becoming $\mathtt{z} \in (0,1)$ if $V \geq 0$ (including the case of pure hard cores, with $V \equiv 0$). The quantity W^{-} is given in Equation (1.1.3a).

To simplify the notation, we omit the indices/arguments $\mathtt{z}$ and β whenever it does not lead to a confusion. A straightforward generalization of the above concepts can be done by including an external potential field induced by a particle configuration $\mathbf{x}(\Lambda^{\mathrm{c}})$ represented by a finite or countable subset in the complement Λ^{c} such that $|y-y'| \geq a$ $\forall$ pair $y, y' \in \mathbf{x}(\Lambda^{\mathrm{c}})$ with $y \neq y'$. Viz., the Hamiltonian $H_{n,\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})}$ is given by

$$\left(H_{n,\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})}\phi_n\right)\left(\underline{x}_1^n\right) = \left(H_{n,\Lambda}\phi_n\right)\left(\underline{x}_1^n\right) + \sum_{1\leq j\leq n}\sum_{y\in\mathbf{x}(\Lambda^{\mathrm{c}})} V\left(|x(j)-y|\right)\phi_n\left(\underline{x}_1^n\right) \tag{1.1.15}$$

and possesses the properties listed above for $H_{n,\Lambda}$. This enables us to introduce the Gibbs operators $G_{n,\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})}$ and $G_{\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})}$, the partition functions $\Xi_n(\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}}))$ and $\Xi(\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}}))$, the DM $R_{\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})}$, the GS $\varphi_{\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})}$ and the RDMs $R^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})}$, $\Lambda_0 \subset \Lambda$. Viz.,

$$\begin{gathered} G_{n,\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})} = \exp\left[-\beta H_{n,\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})}\right], \\ \Xi_n(\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})) := \mathrm{tr}_{\mathrm{L}_2^{\mathrm{sym}}(\Lambda_a^n)} G_{n,\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}}} \in (0,+\infty), \end{gathered} \tag{1.1.16}$$

$$\begin{gathered} G_{\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})} = \bigoplus_{n\geq 0} \mathtt{z}^n G_{n,\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})}, \\ \Xi(\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})) := \sum_{n\geq 0} \mathtt{z}^n \Xi_n(\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})) = \mathrm{tr}_{\mathcal{H}(\Lambda)} G_{\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})} \in (0,+\infty). \end{gathered} \tag{1.1.17}$$

For an empty exterior particle configuration $\mathbf{x}(\Lambda^{\mathrm{c}}) = \emptyset$, the argument $\mathbf{x}(\Lambda^{\mathrm{c}})$ will be omitted. (Although the Hamiltonian $H_{n,\Lambda}$ and its derivatives $G_{n,\Lambda}$, G_Λ and so on, are particular examples of $H_{n,\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})}$, etc., (with $\mathbf{x}(\Lambda^{\mathrm{c}})$ being an empty configuration), we will now and again address this specific example individually, for its methodological significance.)

1.2. The Thermodynamic Limit. The Shift-Invariance Property

The key concept of Statistical Mechanics is the thermodynamic limit; in the context of this work it is $\lim\limits_{\Lambda\nearrow\mathbb{R}^d}$. The quantities and objects established as limiting points in the course of this limit are often referred to as infinite-volume ones (e.g., infinite-volume RDM or GS). The existence and uniqueness of a limiting object is often interpreted as absence of a phase transition, a multitude of such objects (viz., depending on the boundary conditions for the Hamiltonian or the choice of external configuration) is treated as an exhibition of a phase transition, see [3,7,9,18–22]. However, there exists an elegant alternative where infinite-volume values are identified in terms that, at least formally, do not invoke the thermodynamic limit. For classical systems, this is the DLR equations and for the so-called quantum spin systems—the KMS equations. (The latter involves an infinite-volume dynamics which is not affected by phase transitions in terms of GSs.) Unfortunately, the KMS equations are not directly available for the class of quantum systems under consideration in this paper, since the Hamiltonians $H_{n,\Lambda}$ and $H_{n,\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})}$ are not bounded.

In this paper we employ a construction generalizing the classical DLR equation and—in dimension $d=2$—establish shift-invariance property for the emerging objects (the RDMs). Observe that $\forall$ cube $\Lambda_0 (= \Lambda_0(b,\mathrm{L}_0)) = \times_{1\leq j\leq d}\left[\mathtt{b}^j - \mathrm{L}_0, \mathtt{b}^j + \mathrm{L}_0\right]$ centered at $b = \left(\mathtt{b}^1,\ldots,\mathtt{b}^d\right)$ and vector $s = \left(\mathtt{s}^1,\ldots,\mathtt{s}^d\right) \in \mathbb{R}^d$, the Fock spaces $\mathcal{H}(\Lambda_0)$ and $\mathcal{H}(\mathtt{S}(s)\Lambda_0)$ are related through a pair of mutually inverse shift isomorphisms

$$\mathtt{U}^{\Lambda_0}(s):\ \mathcal{H}(\Lambda_0) \to \mathcal{H}(\mathtt{S}(-s)\Lambda_0) \text{ and } \mathtt{U}^{\Lambda_0}(-s):\ \mathcal{H}(\mathtt{S}(s)\Lambda_0) \to \mathcal{H}(\Lambda_0).$$

Here, $\mathtt{S}(\mathtt{s})$ stands for the shift isometry $\mathbb{R}^d \to \mathbb{R}^d$:

$$\mathtt{S}(s)y = y + s, \quad y \in \mathbb{R}^d, \tag{1.2.1}$$

and $\mathtt{S}(s)\Lambda_0$ is for the image of Λ_0:

$$\mathtt{S}(s)\Lambda_0 = \mathop{\times}_{1\le j\le d} \left[\mathtt{b}^j + \mathtt{s}^j - \mathrm{L}^0, \mathtt{b}^j + \mathtt{s}^j + \mathrm{L}^0\right] \; (= \Lambda_0(b+s, \mathrm{L}_0)). \tag{1.2.2}$$

The isomorphisms $\mathtt{U}^{\Lambda_0}(s)$ and $\mathtt{U}^{\Lambda_0}(-s)$ are given by

$$\begin{matrix} \left(\mathtt{U}^{\Lambda_0}(s)\phi_n\right)(\underline{x}_1^n) = \phi_n(\mathtt{S}(-s)\underline{x}_1^n), \\ \left(\mathtt{U}^{\Lambda_0}(-s)\phi_n\right)(\underline{x}_1^n) = \phi_n(\mathtt{S}(s)\underline{x}_1^n), \end{matrix} \quad \underline{x}_1^n \in (\Lambda_0)^n, \tag{1.2.3}$$

where $\phi_n \in \mathrm{L}_2^{\mathrm{sym}}((\Lambda_0)^n)$, $n = 0, 1, \ldots$.

The Fock spaces $\mathcal{H}(\Lambda)$ and $\mathcal{H}(\Lambda_0)$ (see (1.1.9)) can be conveniently represented as $\mathrm{L}_2(\mathcal{C}_a(\Lambda))$ and $\mathrm{L}_2(\mathcal{C}_a(\Lambda^0))$, respectively. Here and below, $\mathcal{C}(\Lambda)$ denotes the collection of finite (unordered) subsets $\mathbf{x} \subset \Lambda$ (including the empty set) with the Lebesgue–Poisson measure

$$\begin{array}{c} \mathrm{d}\mathbf{x} = \dfrac{1}{(\sharp\, \mathbf{x})!} \displaystyle\prod_{x\in\mathbf{x}} \mathrm{d}x, \;\; \sharp\, \mathbf{x} < \infty \;\; \Big(\text{with} \int_{\mathcal{C}(\Lambda)} \mathrm{d}\mathbf{x} = \exp\left[\ell(\Lambda)\right] \\ \text{where } \ell \text{ is the Lebesgue measure on } \mathbb{R}^d\Big), \end{array} \tag{1.2.4}$$

and $\mathcal{C}_a(\Lambda)$ stands for the subset of $\mathcal{C}(\Lambda)$ formed by $\mathbf{x} \subset \Lambda$ with

$$\min \left[|x - x'| : \; x, x' \in \mathbf{x}, \; x \neq x'\right] \geq a. \tag{1.2.5}$$

(The symbol $\sharp$ is used for the cardinality of a given set.) The same meaning is attributed to the notation $\mathcal{C}_a(\mathbb{R}^d)$ and $\mathcal{C}_a(\Lambda^{\mathrm{c}})$ (here, we mean finite or countable sets $\mathbf{x} \subset \mathbb{R}^d$ and $\mathbf{x}' \subset \Lambda^{\mathrm{c}}$, respectively, obeying (1.2.5)).

In Theorem 1 below, we speak of a pair of fixed cubes, $\Lambda_1 \subset \Lambda_0$ where $\Lambda_0 = \mathop{\times}\limits_{1\le j\le d} [\mathtt{b}_0^j - \mathrm{L}_0, \mathtt{b}_0^j + \mathrm{L}_0]$ and $\Lambda_1 = \mathop{\times}\limits_{1\le j\le d} [\mathtt{b}_1^j - \mathrm{L}_1, \mathtt{b}_1^j + \mathrm{L}_1]$, centered at $b_0 = (\mathtt{b}_0^1, \ldots, \mathtt{b}_0^d)$ and $b_1 = (\mathtt{b}_1^1, \ldots, \mathtt{b}_1^d)$. On the other hand, a sequence of boxes $\Lambda(k) = [-\mathrm{L}(k), \mathrm{L}(k)]^d \nearrow \mathbb{R}^d$ is present, of sidelengths $2\mathrm{L}(k) \to \infty$ as $k \to \infty$, which may depend on Λ_1 and Λ_0. We use the term 'box' when referring to a physical volume where a given system is confined and 'cube' while bearing in mind a 'localized' sub-volume as a part of a proof. A box will increase to cover the whole $\mathbb{R}^d$ whereas a cube will be fixed or vary within a restricted range.

Theorem 1. *Suppose that* $\mathrm{z} > 0$ *and* $\beta > 0$ *are given, satisfying condition (1.1.14). For any cube* Λ_0*, the family of RDMs* $\{R^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})}, \Lambda \nearrow \mathbb{R}^d\}$ *is compact in the trace-norm operator topology in* $\mathcal{H}(\Lambda_0)$*, for any choices of particle configurations* $\mathbf{x}(\Lambda^{\mathrm{c}}) \in \mathcal{C}_a(\Lambda^{\mathrm{c}})$*. Any limit-point operator* R^{Λ_0} *for* $\{R^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})}\}$ *is a positive-definite operator in* $\mathcal{H}(\Lambda_0)$ *of trace* 1*. Furthermore, let* $\Lambda_1 \subset \Lambda_0$ *be a pair of cubes and* R^{Λ_1}*,* R^{Λ_0} *be a pair of limit-point RDMs such that*

$$R^{\Lambda_1} = \lim_{k\to+\infty} R^{\Lambda_1}_{\Lambda(k)|\mathbf{x}(\Lambda(k)^{\mathrm{c}}} \text{ and } R^{\Lambda_0} = \lim_{k\to+\infty} R^{\Lambda_0}_{\Lambda(k)|\mathbf{x}(\Lambda(k)^{\mathrm{c}}} \tag{1.2.6}$$

for a sequence of boxes $\Lambda(k) \nearrow \mathbb{R}^d$ *and external configurations* $\mathbf{x}(\Lambda(k)^{\mathrm{c}})$ *obeying (1.2.5). Then* R^{Λ_1} *and* R^{Λ_0} *satisfy the compatibility property*

$$R^{\Lambda_1} = \mathrm{tr}_{\mathcal{H}(\Lambda_0\setminus\Lambda_1)} R^{\Lambda_0}. \tag{1.2.7}$$

The next theorem is established in dimension $d = 2$.

Theorem 2. *Set $d = 2$ and assume that V has a finite range, with $V^{(1)}, V^{(2)} < \infty$; cf. (1.1.4). Let $\mathtt{z} > 0$ and $\beta > 0$ be such that Equation (1.1.14) is satisfied. Given a square $\Lambda_0 = [\mathtt{b}^1 - \mathrm{L}_0, \mathtt{b}^1 + \mathrm{L}_0] \times [\mathtt{b}^2 - \mathrm{L}_0, \mathtt{b}^2 + \mathrm{L}_0]$ and a vector $s = (\mathtt{s}^1, \mathtt{s}^2)$, consider limit-point RDMs R^{Λ_0} and $R^{\mathtt{S}(\mathtt{s})\Lambda_0}$ such that*

$$R^{\mathtt{S}(\mathtt{s})\Lambda_0} = \lim_{k\to+\infty} R^{\mathtt{S}(\mathtt{s})\Lambda_0}_{\Lambda(k)|\mathbf{x}(\Lambda(k)^{\mathrm{c}})} \text{ and } R^{\Lambda_0} = \lim_{k\to+\infty} R^{\Lambda_0}_{\Lambda(k)|\mathbf{x}(\Lambda(k)^{\mathrm{c}})} \tag{1.2.8}$$

for a sequence of boxes $\Lambda(k) = [-\mathrm{L}(k), \mathrm{L}(k)]^2 \nearrow \mathbb{R}^2$ and external configurations $\mathbf{x}(\Lambda(k)^{\mathrm{c}}) \in \mathcal{C}_a(\Lambda^{\mathrm{c}}(k))$. Then R^{Λ_0} and $R^{\mathtt{S}(\mathtt{s})\Lambda_0}$ have the property that

$$R^{\mathtt{S}(\mathtt{s})\Lambda_0} = \mathtt{U}^{\Lambda_0}(-s) R^{\Lambda_0} \mathtt{U}^{\Lambda_0}(s). \tag{1.2.9}$$

In the future, the bound (1.1.14) will be assumed without stressing it every time again. Also, referring to external configurations $\mathbf{x}(\Lambda^{\mathrm{c}})$ and $\mathbf{x}(\Lambda(k)^{\mathrm{c}})$, we always assume that $\mathbf{x}(\Lambda^{\mathrm{c}}) \in \mathcal{C}_a(\Lambda^{\mathrm{c}})$ and $\mathbf{x}(\Lambda(k)^{\mathrm{c}}) \in \mathcal{C}_a(\Lambda^{\mathrm{c}}(k))$.

A direct corollary of Theorem 1 is the construction of a limit-point Gibbs state. To this end, it suffices to consider a countable collection of cubes $\Lambda_0 (= \Lambda_0(b_0, \mathrm{L}_0)) = \underset{1\le j\le d}{\times} [\mathtt{b}_0^j - \mathrm{L}_0, \mathtt{b}_0^j + \mathrm{L}_0]$, with rational $b_0 = (\mathtt{b}_0^1, \ldots, \mathtt{b}_0^d)$ and L_0. By invoking a diagonal process, we can guarantee that, as $\Lambda \nearrow \mathbb{R}^d$, given any family of external configurations $\mathbf{x}(\Lambda^{\mathrm{c}})$, one can extract a sequence $\Lambda(k) \nearrow \mathbb{R}^d$ such that (i) $\forall$ cube Λ_0 from the collection, $\exists$ the trace-norm limit

$$R^{\Lambda_0} = \lim_{k\to+\infty} R^{\Lambda_0}_{\Lambda(k)|\mathbf{x}(\Lambda(k)^{\mathrm{c}})} \tag{1.2.10}$$

and (ii) the limiting operators relation (1.2.7) holds true $\forall$ pair of cubes from the collection, $\Lambda_1 = \underset{1\le j\le d}{\times} [\mathtt{b}_1^j - \mathrm{L}_1, \mathtt{b}_1^j + \mathrm{L}_1]$ and $\Lambda_0 = \underset{1\le j\le d}{\times} [\mathtt{b}_0^j - \mathrm{L}_0, \mathtt{b}_0^j + \mathrm{L}_0]$ whenever $\Lambda_1 \subset \Lambda_0$. This enables us to define an infinite-volume Gibbs state φ by setting

$$\varphi(A) = \mathrm{tr}_{\mathcal{H}(\Lambda_0)}\big(A R^{\Lambda_0}\big), \quad A \in \mathfrak{B}(\Lambda_0), \tag{1.2.11}$$

for any cube $\Lambda_0 \subset \mathbb{R}^d$. More precisely, φ is a state of the quasilocal C*-algebra $\mathfrak{B}(\mathbb{R}^d)$ defined as the norm-closure of the inductive limit $\mathfrak{B}^0(\mathbb{R}^d)$:

$$\mathfrak{B}(\mathbb{R}^d) = \text{norm-closure}\left[\mathfrak{B}^0(\mathbb{R}^d)\right], \quad \mathfrak{B}^0(\mathbb{R}^d) = \operatorname*{ind\,lim}_{\Lambda\nearrow\mathbb{R}^d} \mathfrak{B}(\Lambda). \tag{1.2.12}$$

A corollary of Theorem 2 is

Theorem 3. *For $d = 2$, suppose that the conditions of Theorem 2 are fulfilled. Then any limit-point Gibbs state φ is shift-invariant:*

$$\varphi(A) = \varphi(\mathtt{S}(s)A), \quad A \in \mathfrak{B}(\mathbb{R}^2). \tag{1.2.13}$$

Here, $\mathtt{S}(s)A)$ stands for the shift of the argument A: if $A \in \mathfrak{B}(\Lambda_0)$ then

$$\mathtt{S}(s)A = \mathtt{U}^{\Lambda_0}(s) A \mathtt{U}^{\Lambda_0}(-s) \in \mathfrak{B}(\mathtt{S}(s)\Lambda_0). \tag{1.2.14}$$

1.3. Integral Kernels of Gibbs Operators and RDMs

According to the adopted realization of the Fock space $\mathcal{H}(\Lambda)$ as $\mathrm{L}_2(\mathcal{C}_a(\Lambda))$, its elements are represented by functions $\phi_\Lambda : \ \mathbf{x}(\Lambda) \in \mathcal{C}_a(\Lambda) \mapsto \phi_\Lambda(\mathbf{x}(\Lambda)) \in \mathbb{C}$, with

$$\int_{\mathcal{C}(\Lambda)} |\phi_\Lambda(\mathbf{x}(\Lambda))|^2 \, \mathrm{d}\mathbf{x}(\Lambda) < \infty. \tag{1.3.1}$$

The space $\mathcal{H}(\Lambda_0)$ is described in a similar manner: here, we will use a short-hand notation $\mathbf{x}_0$ and $\mathbf{y}_0$ instead of $\mathbf{x}(\Lambda_0), \mathbf{y}(\Lambda_0) \in \mathcal{C}_a(\Lambda_0)$.

The first step in the proof of Theorems 1 and 2 is to reduce their assertions to statements about the integral kernels $F_\Lambda^{\Lambda_0}$, $F_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}$ and F^{Λ_0} for the RDMs $R_\Lambda^{\Lambda_0}$, $R_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}$ and their infinite-volume counterpart R^{Λ_0}; we call these kernels RDMKs for short. Indeed, $R_\Lambda^{\Lambda_0}$, $R_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}$ and R^{Λ_0} are integral operators:

$$\left(R_\Lambda^{\Lambda_0}\phi_\Lambda\right)(\mathbf{x}_0) = \int_{\mathcal{C}_a(\Lambda)} F_\Lambda^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0)\phi_\Lambda(\mathbf{y}_0)\mathrm{d}\mathbf{y}_0, \tag{1.3.2}$$

$$\left(R_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}\phi_\Lambda\right)(\mathbf{x}_0) = \int_{\mathcal{C}_a(\Lambda)} F_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0)\phi_\Lambda(\mathbf{y}_0)\mathrm{d}\mathbf{y}_0 \tag{1.3.3}$$

and

$$\left(R^{\Lambda_0}\phi_\Lambda\right)(\mathbf{x}_0) = \int_{\mathcal{C}_a(\Lambda)} F^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0)\phi_\Lambda(\mathbf{y}_0)\mathrm{d}\mathbf{y}_0. \tag{1.3.4}$$

The RDMKs $F_\Lambda^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0)$ and $F_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0)$ – and ultimately $F^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0)$—admit a Feynman–Kac (FK) representation providing a basis for future analysis. Here, we state properties of these kernels in Theorems 4 and 5:

Theorem 4. *Under the conditions of Theorem 1, for any cube Λ_0 and for any choice of particle configurations $\mathbf{x}(\Lambda^c) \in \mathcal{C}_a(\Lambda^c)$, the family of RDMKs $F_{\Lambda|\mathbf{x}(\Lambda^c)}^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0)$ is compact in the space of continuous functions $C^0(\mathcal{C}_a(\Lambda_0) \times \mathcal{C}_a(\Lambda_0))$. Any limit-point function*

$$(\mathbf{x}_0, \mathbf{y}_0) \in \mathcal{C}_a(\Lambda_0) \times \mathcal{C}_a(\Lambda_0) \mapsto F^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0) \tag{1.3.5}$$

determines a positive-definite operator R^{Λ_0} in $\mathcal{H}(\Lambda_0)$ of trace 1 (a limit-point RDM). Furthermore, let $\Lambda_1 \subset \Lambda_0$ be a pair of squares and F^{Λ_1}, F^{Λ_0} a pair of limit-point RDMKs such that

$$F^{\Lambda_1} = \lim_{k\to+\infty} F_{\Lambda(k)}^{\Lambda_1} \text{ and } F^{\Lambda_0} = \lim_{k\to+\infty} F_{\Lambda(k)|\mathbf{x}(\Lambda(k)^c)}^{\Lambda_0} \tag{1.3.6}$$

in $C^0(\mathcal{C}_a(\Lambda_0) \times \mathcal{C}_a(\Lambda_0))$ for a sequence of squares $\Lambda(k) \nearrow \mathbb{R}^2$, boundary conditions on $\partial^{\rm out}\Lambda(k)^*$ and external configurations $\mathbf{x}(\Lambda(k)^c)$. Then the corresponding limit-point RDMs R^{Λ_1} and R^{Λ_0} obey (1.2.7).

Theorem 4 implies Theorem 1 with the help of Lemma 1.5 from [23] (going back to Lemma 1 in [24]). In turn, Theorem 2 is a direct corollary of

Theorem 5. *Set $d = 2$ and assume the conditions of Theorem 2. Given a square Λ_0 and a vector $s = (\mathrm{s}^1, \mathrm{s}^2)$, consider limit-point RDMKs F^{Λ_0} and $F^{\mathrm{S}(s)\Lambda_0}$ such that*

$$F^{\mathrm{S}(s)\Lambda_0} = \lim_{k\to+\infty} F_{\Lambda(k)|\mathbf{x}(\Lambda(k)^c)}^{\mathrm{S}(s)\Lambda_0} \text{ and } F^{\Lambda_0} = \lim_{k\to+\infty} F_{\Lambda(k)|\mathbf{x}(\Lambda(k)^c)}^{\Lambda_0} \tag{1.3.7}$$

for a sequence of squares $\Lambda(k) \nearrow \mathbb{R}^2$, boundary conditions on $\partial^{\rm out}\Lambda(k)^*$ and external configurations $\mathbf{x}(\Lambda(k)^c)$. Then, $\forall\ \mathbf{x}_0, \mathbf{y}_0 \in \mathcal{C}_a(\Lambda_0)$ and $s = \left(\mathrm{s}^1, \mathrm{s}^2\right) \in \mathbb{R}^2$,

$$F^{\mathrm{S}(s)\Lambda_0}(\mathrm{S}(s)\mathbf{x}_0, \mathrm{S}(s)\mathbf{y}_0) = F^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0). \tag{1.3.8}$$

Therefore, we focus on the proof of Theorems 4 and 5. In fact, we will establish the properties for more general objects—FK-DLR functionals.

2. The FK Representation and the FK-DLR Equation

2.1. The Background of the FK-Representation

We begin with Definitions 1–3 used in Lemma 1 below.

Definition 1 (Path spaces). *As above, x, y stand for points in $\mathbb{R}^d$, $\underline{x} = \underline{x}_1^n = \{x(1), \ldots, x(n)\}$ and $\underline{y} = \underline{y}_1^n = \{y(1), \ldots, y(n)\}$ for points in Λ^n. Next, $\gamma = \gamma_n$ denotes a permutation of the nth order, $\gamma\underline{y} = \{y(\gamma(1)), \ldots, y(\gamma(n))\}$ stands for the vector with permuted entries and $\mathbf{x}(\Lambda)$ for a point in $\mathcal{C}(\Lambda)$ (i.e., a finite subset of Λ). Furthermore, we will use the following system of notation:*

(i) $\overline{\mathcal{W}}^{k\beta}(x, y)$—the space of continuous paths $\overline{\omega} = \overline{\omega}_{x,y} : [0, k\beta] \to \mathbb{R}^d$ of time-length $k\beta$ (the parameter k is called the time-length multiplicity), with $\overline{\omega}(0) = x$, $\overline{\omega}(k\beta) = y$, where $k = 1, 2, \ldots$.

(ii) $\overline{\mathcal{W}}^*(x, y) = \bigcup\limits_{k \geq 1} \overline{\mathcal{W}}^{k\beta}(x, y)$ – the space of continuous paths $\overline{\omega}^* = \overline{\omega}^*_{x,y} : [0, \beta] \to \mathbb{R}^d$ of a variable time-length $k\beta$, with $\overline{\omega}^*(0) = x$, $\overline{\omega}^*(k\beta) = y$. In the future, we set: $k(\overline{\omega}^*) = k$ when $\overline{\omega}^* \in \overline{\mathcal{W}}^{k\beta}(x, y)$.

(iii) $\mathcal{W}^*(x) = \overline{\mathcal{W}}^*(x, x)$—the space of loops (closed paths) $\omega^* = \omega^*_x$ with $\omega^*(0) = \omega^*(\beta) = x$.

(iv) $\overline{\mathcal{W}}^*(\underline{x}_1^n, \underline{y}_1^n) = \mathop{\times}\limits_{1 \leq j \leq n} \overline{\mathcal{W}}^*(x(j), y(j))$ – the space of (ordered) path collections $\overline{\Omega}^* = \{\overline{\omega}^*(1), \ldots, \overline{\omega}^*(n)\}$ where $\overline{\omega}^*(j) \in \overline{\mathcal{W}}^*(x(j), y(j))$; for $\underline{x}_1^n = \{x(1), \ldots, x(n)\}$, $\underline{y}_1^n = \{y(1), \ldots, y(n)\}$, $n = 1, 2, \ldots$.

(v) $\underline{\mathcal{W}}^*(\underline{x}_1^n, \underline{y}_1^n) = \bigcup\limits_{\gamma_n} \overline{\mathcal{W}}^*(\underline{x}_1^n, \gamma_n \underline{y}_1^n)$—the space of path collections $\underline{\Omega}^* = \{\overline{\omega}^*(1), \ldots, \overline{\omega}^*(n)\}$ with permuted endpoints (that is, with loops $\overline{\omega}^*(j) \in \overline{\mathcal{W}}^*(x(j), y(\gamma_n j))$, $1 \leq j \leq n$), where $\gamma_n \underline{y}_1^n = \{y(\gamma_n 1), \ldots, y(\gamma_n n)\}$ and γ_n is a permutation of order n. Alternatively, $\underline{\mathcal{W}}^*(\mathbf{x}, \mathbf{y}) = \bigcup\limits_{\gamma : \mathbf{x} \leftrightarrow \mathbf{y}} \overline{\mathcal{W}}^*(x, \gamma x)$ where $\mathbf{x}, \mathbf{y} \in \mathcal{C}_a(\mathbb{R}^d)$ with $\sharp\,\mathbf{x} = \sharp\,\mathbf{y}$ and $\gamma : \mathbf{x} \leftrightarrow \mathbf{y}$ means γ is a one-to-one map between $\mathbf{x}$ and $\mathbf{y}$. Hence, $\underline{\Omega}^* \in \underline{\mathcal{W}}^*(\mathbf{x}, \mathbf{y})$ is a path collection $\{\overline{\omega}^*(x, y) : x \in \mathbf{x}, y = \gamma x \in \mathbf{y}\}$ where $\gamma : \mathbf{x} \leftrightarrow \mathbf{y}$.

(vi) $\mathcal{W}^*(\mathbf{x}) = \mathop{\times}\limits_{x \in \mathbf{x}} \mathcal{W}^*(x)$ – the space of loop collections $\Omega^*(\mathbf{x}) = \{\omega^*(x), x \in \mathbf{x}\}$ (Ω^* for short) with a given (finite) initial/end-point particle configuration $\mathbf{x} \in \mathcal{C}_a(\mathbb{R}^d)$, where $\omega(x) \in \mathcal{W}^*(x)$.

(vii) $\mathcal{W}^*(\Lambda) = \bigcup\limits_{\mathbf{x} \in \mathcal{C}_a(\Lambda)} \mathcal{W}^*(\mathbf{x})$ – the space of loop collections $\Omega^* = \Omega^*(\Lambda) = \{\omega^*(x),\ x \in \mathbf{x}\}$ with various initial/end-point configurations $\mathbf{x} = \mathbf{x}(\Lambda) \in \mathcal{C}_a(\Lambda)$. Sometimes it will be helpful to stress that an element $\Omega^* \in \mathcal{W}^*(\Lambda)$ is a pair $\left[\mathbf{x}(\Lambda), \Omega^*(\mathbf{x}(\Lambda))\right]$, where $\Omega^*(\mathbf{x}(\Lambda)) \in \mathcal{W}^*(\mathbf{x}(\Lambda))$, and treat a loop $\omega^*(x) \in \mathcal{W}^*_x$ (or rather its shift $\mathtt{S}(-x)\omega^*(x) \in \mathcal{W}^*(0)$) as a 'mark' for point $x \in \mathbf{x}(\Lambda)$. (Here and below, the loop $\mathtt{S}(s)\omega^*$ is defined by $(\mathtt{S}(s)\omega^*)(\mathtt{t}) = \omega^*(\mathtt{t}) + s$, $s \in \mathbb{R}^d$, $\mathtt{t} \in [0, \beta k(\omega^*)]$.) Such a view is useful when we work with probability measures (PMs) on $\mathcal{W}^*(\Lambda)$: in the probabilistic terminology these PMs represent d-dimensional random marked point processes (RMPPs) in Λ with marks from $\mathcal{W}^*(0)$, the space of loops starting and finishing at 0.

An element $\overline{\Omega}^*$ from $\overline{\mathcal{W}}^*(\underline{x}, \underline{y})$ is called a path collection/configuration, with the initial/terminal particle configurations $\underline{x}$, $\underline{y}$. (For simplicity, we write $\underline{x}$ and $\underline{y}$ instead of $\underline{x}_1^n$ and $\underline{y}_1^n$.) The same term is used for $\underline{\Omega}^* \in \underline{\mathcal{W}}^*(\mathbf{x}, \mathbf{y})$. Likewise, an element $\Omega^* \in \mathcal{W}^*(\Lambda)$ is called a loop configuration over Λ; if $\Omega^* \in \mathcal{W}^*(\mathbf{x}(\Lambda))$, we say that $\mathbf{x}(\Lambda)$ is the initial particle configuration for Ω^*. The time-length multiplicity of a path $\overline{\omega}^* \in \overline{\mathcal{W}}_{x,y}$ is denoted by $k(\overline{\omega}^*)$. The next series of definitions is introduced for a fixed $\mathtt{t} \in [0, \beta]$. Namely, given a path $\overline{\omega}^* \in \overline{\mathcal{W}}_{x,y}$, we call the set

$$\{\overline{\omega}^*(\bar{l}\beta + \mathtt{t}),\ \bar{l} = 0, \ldots, k(\overline{\omega}^*) - 1\} \subset \mathbb{R}^d$$

the $\mathtt{t}$-section of $\overline{\omega}^*$ and denote it by $\{\overline{\omega}^*\}(\mathtt{t})$. Next, given a path collection $\overline{\Omega}^* = \{\omega^*(1), \ldots, \omega^*(n)\} \in \overline{\mathcal{W}}^*_{\underline{x},\underline{y}}$, the $\mathtt{t}$-section for $\overline{\Omega}^*$ is defined as the union

$$\{\overline{\Omega}^*\}(\mathtt{t}) = \bigcup_{1 \leq j \leq n} \overline{\omega}^*(j, \mathtt{t})$$

where $\{\overline{\omega}^*(j)\}(\mathtt{t}) = \{\overline{\omega}^*(j, \bar{l}(j)\beta + \mathtt{t}),\ 0 \leq \bar{l}(j) < k(\overline{\omega}^*(j))\}$ *is the* $\mathtt{t}$*-section for path* $\overline{\omega}(j) \in \overline{\Omega}^*$ *(thus,* $\{\overline{\Omega}^*\}(\mathtt{t})$ *again is a subset of* $\mathbb{R}^d$*). Likewise, given a loop configuration* $\Omega^* = \{\omega^*(x), x \in \mathbf{x}(\Lambda)\} \in \mathcal{W}^*(\mathbf{x}(\Lambda))$*, the set*

$$\{\Omega^*\}(\mathtt{t}) = \bigcup_{1\leq j\leq n} \{\omega^*(x)\}(\mathtt{t}) \subset \mathbb{R}^d$$

is called the $\mathtt{t}$*-section of* $\Omega^*(\mathtt{t})$*. Here,* $\{\omega^*(x)\}(\mathtt{t}) = \{\omega^*(x, l\beta + \mathtt{t}),\ 0 \leq l < k(\omega^*(x)),\ x \in \mathbf{x}(\Lambda)\} \subset \mathbb{R}^d$ *is the* $\mathtt{t}$*-section of loop* $\omega^*(x)$*.*

Similar definitions and terms will be used for a square $\Lambda_0 \subset \Lambda$ *or the set-theoretical difference* $\Lambda \setminus \Lambda_0$*.*

All path/loop spaces $\overline{\mathcal{W}}^{k\beta}(x,y)$, $\overline{\mathcal{W}}^*(x,y)$, $\mathcal{W}^*(x)$, $\overline{\mathcal{W}}^*(\underline{x},\underline{y})$, $\underline{\mathcal{W}}^*(\underline{x},\underline{y})$, $\mathcal{W}^*(\mathbf{x},\mathbf{y})$, $\mathcal{W}^*(\mathbf{x})$, $\mathcal{W}^*(\Lambda)$ *from (i)–(vii) contain subsets* $\overline{\mathcal{W}}_a^{k\beta}(x,y)$, $\overline{\mathcal{W}}_a^*(x,y)$, $\mathcal{W}_a^*(x)$, $\overline{\mathcal{W}}_a^*(\underline{x},\underline{y})$, $\underline{\mathcal{W}}_a^*(\underline{x},\underline{y})$, $\mathcal{W}_a^*(\mathbf{x},\mathbf{y})$, $\mathcal{W}_a^*(\mathbf{x})$ *and* $\mathcal{W}_a^*(\Lambda)$ *extracted by the condition that* $\forall\ \mathtt{t} \in [0,\beta]$ *no two distinct points in the t-section lie at a Euclidean distance* $\leq a$*. In other words, all sections* $\{\overline{\omega}^*\}(\mathtt{t})$, $\{\overline{\Omega}^*\}(\mathtt{t})$ $\{\Omega^*\}(\mathtt{t})$ *are (finite) particle configurations lying in* $\mathcal{C}_a(\mathbb{R}^d)$*.*

Definition 2 (Path measures)**.** *The spaces introduced in Definition 1 are equipped with standard sigma-algebras (generated by cylinder subsets and operations on them), see [25]. We consider various measures on these sigma-algebras:*

(i) $\overline{\mathbb{P}}_{x,y}^{k\beta}$*—the (non-normalized) measure on* $\overline{\mathcal{W}}^{k\beta}(x,y)$ *(the Wiener bridge of time-length* $k\beta$*), with* $\overline{\mathbb{P}}_{x,y}^{k\beta}\left(\overline{\mathcal{W}}^{k\beta}(x,y)\right) = (2\pi k\beta)^{-d/2} \exp\left[-|x-y|^2/(2k\beta)\right]$*.*

(ii) $\overline{\mathbb{P}}_{x,y}^*$*—the sum-measure* $\sum_{k\geq 1} \overline{\mathbb{P}}_{x,y}^{k\beta}$ *on* $\overline{\mathcal{W}}^*(x,y)$*.*

(iii) $\mathbb{P}_x^* = \overline{\mathbb{P}}_{x,x}^*$*—the sum-measure* $\sum_{k\geq 1} \overline{\mathbb{P}}_x^{k\beta}$ *on* $\mathcal{W}^*(x)$*.*

(iv) $\overline{\mathbb{P}}_{\underline{x},\underline{y}}^* = \mathop{\times}_{1\leq j\leq n} \overline{\mathbb{P}}_{x(j),y(j)}^*$*—the product-measure on* $\overline{\mathcal{W}}^*(\underline{x},\underline{y})$ *(a vector Wiener bridge) under which the components* $\overline{\omega}^*(j) \in \overline{\mathcal{W}}^*(x(j),y(j))$ *are independent.*

(v) $\underline{\mathbb{P}}_{\underline{x},\underline{y}}^* = \sum_{\gamma_n} \overline{\mathbb{P}}_{\underline{x},\gamma_n\underline{y}}^*$ *and* $\underline{\mathbb{P}}_{\mathbf{x},\mathbf{y}}^* = \sum_{\gamma:\mathbf{x}\leftrightarrow\mathbf{y}} \overline{\mathbb{P}}_{\mathbf{x},\gamma\mathbf{x}}^*$*—the sum-measures on* $\underline{\mathcal{W}}^*(\underline{x},\underline{y})$ *and* $\underline{\mathcal{W}}^*(\mathbf{x},\mathbf{y})$*. Here* $\overline{\mathbb{P}}_{\underline{x},\gamma_n\underline{y}}^* = \mathop{\times}_{1\leq j\leq n} \overline{\mathbb{P}}_{x(j),y(\gamma_n j)}^*$ *and* $\overline{\mathbb{P}}_{\mathbf{x},\gamma\mathbf{y}}^* = \mathop{\times}_{x\in\mathbf{x}} \overline{\mathbb{P}}_{x,y}^*$ *where* $y = \gamma x$*.*

(vi) $\mathbb{P}_{\mathbf{x}}^* = \mathop{\times}_{x\in\mathbf{x}} \mathbb{P}_x^*$*—the product-measure on* $\mathcal{W}^*(\mathbf{x})$*.*

(vii) $\mathrm{d}\Omega^*(\Lambda) = \mathrm{d}\mathbf{x}(\Lambda) \times \mathbb{P}_{\mathbf{x}(\Lambda)}^*(\mathrm{d}\Omega^*)$*—the measure on* $\mathcal{W}^*(\Lambda)$ *where* $\mathrm{d}\mathbf{x}(\Lambda)$ *is the Lebesgue–Poisson measure on* $\mathcal{C}(\Lambda)$ *(cf. (1.3.2)). We will use the name Lebesgue–Poisson–Wiener measure (LPWM). Sometimes we will write* $\mathrm{d}^\Lambda\mathbf{x}(\Lambda)$ *and* $\mathrm{d}^\Lambda\Omega^*(\Lambda)$ *in order to stress the dependence upon* Λ*.*

As a rule, we will be working with restrictions of the above measures upon the corresponding subsets $\overline{\mathcal{W}}_a^{k\beta}(x,y)$, $\overline{\mathcal{W}}_a^*(x,y)$, $\mathcal{W}_a^*(x)$, $\overline{\mathcal{W}}_a^*(\underline{x},\underline{y})$, $\underline{\mathcal{W}}_a^*(\underline{x},\underline{y})$, $\underline{\mathcal{W}}_a^*(\mathbf{x},\mathbf{y})$, $\mathcal{W}_a^*(\mathbf{x})$, and $\mathcal{W}_a^*(\Lambda)$.

The Brownian (or Wiener) bridge on the time interval $k\beta$ with the endpoints at 0 is usually defined as a process of the form $\widetilde{W}(\mathtt{t}) = W(\mathtt{t}) - \frac{t}{k\beta}W(k\beta)$, $0 \leq \mathtt{t} \leq k\beta$, where $W(\mathtt{t})$ is a standard Brownian motion; cf., e.g., [26,27]. It is a (non-homogeneous) Markov process, with a strong Markov property that $\forall$ Markov stopping time $\mathcal{T}$, the behavior of the process before time $\mathcal{T} \wedge k\beta$ and after are conditionally independent, given $W(\mathcal{T} \wedge k\beta)$. The Brownian bridge with the initial points at x and the final point at y is constructed as $\widetilde{W}(\mathtt{t}) + x + \mathtt{t}(y - x)$.

Definition 3 (Energy-related functionals)**.** *Given a path* $\overline{\omega}^* \in \overline{\mathcal{W}}_a^*(x,y)$*, we set:*

$$h(\overline{\omega}^*) = \int_0^\beta \mathrm{d}\mathtt{t} \sum_{0\leq l<l'<k(\overline{\omega}^*)} V\left(|\overline{\omega}^*(\mathtt{t}+l\beta) - \overline{\omega}^*(\mathtt{t}+l'\beta)|\right) = \int_0^\beta \mathrm{d}\mathtt{t} E\big(\{\overline{\omega}^*\}(\mathtt{t})\big) \tag{2.1.1}$$

where, for a given finite particle configuration $\mathbf{z} \in \mathcal{C}_a(\mathbb{R}^d)$*, we set:*

$$E(\mathbf{z}) = \frac{1}{2} \sum_{(z,z') \in \mathbf{z} \times \mathbf{z}} V(|z - z'|). \tag{2.1.2}$$

The quantity $h(\overline{\omega})$ *can be interpreted as an energy of path* $\overline{\omega}$*.*

The energy of interaction between two paths, $\overline{\omega}^* \in \overline{\mathcal{W}}_a^*(x,y)$ *and* $\overline{\omega}^{*\prime} \in \overline{\mathcal{W}}_a^*(x',y')$*, is determined by*

$$\begin{aligned} h(\overline{\omega}^*, \overline{\omega}^{*\prime}) \quad &= \int_0^\beta \mathrm{dt} \sum_{0 \le l < k(\overline{\omega}^*)} \sum_{0 \le l' < k(\overline{\omega}^{*\prime})} V\left(\left|\overline{\omega}^*(t + l\beta) - \overline{\omega}^{*\prime}(t + l'\beta)\right|\right) \\ &= \int_0^\beta \mathrm{dt} E\left(\{\overline{\omega}^*\}(\mathrm{t}) \big\| \{\overline{\omega}^{*\prime}\}(\mathrm{t})\right). \end{aligned} \tag{2.1.3}$$

Here, for a given pair of particle configurations $\mathbf{z}, \mathbf{z}' \in \mathcal{C}_a(\mathbb{R}^d)$*, such that* $\mathbf{z} \cup \mathbf{z}' \in \mathcal{C}_a(\mathbb{R}^d)$*,* $\mathbf{z} \cap \mathbf{z}' = \emptyset$ *and at least one of them is finite, we set:*

$$E(\mathbf{z}||\mathbf{z}') = \sum_{(z,z') \in \mathbf{z} \times \mathbf{z}'} V(|z - z'|). \tag{2.1.4}$$

The Definitions 1 and 3 hold for loops as well, obviously.

Next, for a path collection $\underline{\Omega}^* = \{\overline{\omega}^*(1), \ldots, \overline{\omega}^*(n)\} \in \underline{\mathcal{W}}_a^*(\underline{x}, \underline{y})$ *and a loop configuration* $\Omega^* = \{\omega^*(x)\} \in \mathcal{W}_a^*(\mathbf{x}(\Lambda))$*, the energy* $h(\underline{\Omega}^*)$ *of* $\underline{\Omega}^*$ *and the energy* $h(\Omega^*)$ *of* Ω^* *are defined as*

$$h(\underline{\Omega}^*) = \sum_{1 \le j \le n} h(\overline{\omega}^*(j)) + \sum_{1 \le j < j' \le n} h(\overline{\omega}^*(j), \overline{\omega}^*(j')) \tag{2.1.5}$$

and

$$h(\Omega^*) = \sum_{x \in \mathbf{x}(\Lambda)} h(\omega^*(x)) + \frac{1}{2} \sum_{x,x' \in \mathbf{x}(\Lambda):\, x \ne x'} h(\overline{\omega}^*(x), \overline{\omega}^*(x')). \tag{2.1.6}$$

We will also need the energy for various combined collections of paths, loops and particle configurations. Viz., for $\underline{\Omega}^* = \{\overline{\omega}^*(1), \ldots, \overline{\omega}^*(n)\} \in \underline{\mathcal{W}}_a^*(\underline{x}, \underline{y})$ *where* $\underline{x}, \underline{y} \in \Lambda^n$ *and* $\Omega^* = \{\omega^*(x)\} \in \mathcal{W}_a^*(\Lambda)$*,*

$$h(\underline{\Omega}^* \vee \Omega^*) = h(\underline{\Omega}^*) + h(\Omega^*) + h(\underline{\Omega}^*||\Omega^*) \tag{2.1.7}$$

where

$$h(\underline{\Omega}^*||\Omega^*) = \sum_{1 \le j \le n} h(\overline{\omega}^*(j)||\Omega^*) = \int_0^\beta \mathrm{dt} E(\underline{\Omega}^*(\mathrm{t})||\Omega^*(\mathrm{t})). \tag{2.1.8}$$

Finally, for $\mathbf{x}(\Lambda^c) \in \mathcal{C}_a(\Lambda^c)$*,*

$$h(\underline{\Omega}^* \vee \Omega^* | \mathbf{x}(\Lambda^c)) = h(\underline{\Omega}^* \vee \Omega^*) + h(\underline{\Omega}^* \vee \Omega^* || \mathbf{x}(\Lambda^c)) \tag{2.1.9}$$

where

$$h(\underline{\Omega}^* \vee \Omega^* || \mathbf{x}(\Lambda^c)) = \int_0^\beta E(\{\underline{\Omega}^*\}(\mathrm{t}) \cup \{\Omega^*\}(\mathrm{t}) || \mathbf{x}(\Lambda^c)) \mathrm{dt}. \tag{2.1.10}$$

Finally, we introduce functionals K, L and α_Λ*, for path and loop configurations :*

$$K(\underline{\Omega}^*) = \sum_{\overline{\omega}^* \in \underline{\Omega}^*} k(\overline{\omega}^*), \quad K(\Omega^*) = \sum_{\omega^* \in \Omega^*} k(\omega^*), \tag{2.1.11}$$

and

$$L(\Omega^*) = \prod_{\omega^* \in \Omega^*} k(\omega^*). \tag{2.1.12}$$

Here and below,

$$k(\omega^*) = k \text{ when } \omega^* \in \mathcal{W}^{k\beta}(x,x),\ k(\overline{\omega}^*) = k \text{ when } \overline{\omega}^* \in \overline{\mathcal{W}}^{k\beta}(x,y). \tag{2.1.13}$$

The presence of Dirichlet's boundary conditions is manifested in the indicators

$$\alpha_\Lambda(\underline{\Omega}^*) = \prod_{\overline{\omega}^* \in \underline{\Omega}^*} \alpha_\Lambda(\overline{\omega}^*),\ \ \alpha_\Lambda(\Omega^*) = \prod_{\omega^* \in \Omega^*} \alpha_\Lambda(\omega^*), \tag{2.1.14}$$

where

$$\alpha_\Lambda(\overline{\omega}^*) = \mathbf{1}\Big(\overline{\omega}^*(\mathtt{t}) \in \Lambda\ \ \forall\, \mathtt{t} \in \big[0, k(\overline{\omega}^*)\beta\big]\Big). \tag{2.1.15}$$

2.2. The FK-Representation in a Box

As follows from well-known results about the operators $H_{n,\Lambda}$ and $H_{n,\Lambda|\mathbf{x}(\Lambda^c)}$ (see, e.g., [1,20,21]), we have the following properties listed in Lemmas 1 and 2

Lemma 1. *For an external particle configuration $\mathbf{x}(\Lambda^c)$ defining the self-adjoint operators $H_{n,\Lambda|\mathbf{x}(\Lambda^c)}$, the partition function $\Xi\big[\Lambda|\mathbf{x}(\Lambda^c)\big]$ (see (1.1.21)) admits the following representation:*

$$\Xi\big[\Lambda|\mathbf{x}(\Lambda^c)\big] = \int\limits_{\mathcal{W}_a^*(\Lambda)} \alpha_\Lambda(\Omega_\Lambda^*) \frac{\mathbf{z}^{K(\Omega_\Lambda^*)}}{L(\Omega_\Lambda^*)} \exp\Big[-h\Big(\Omega_\Lambda^*|\mathbf{x}(\Lambda^c)\Big)\Big]\, \mathrm{d}^\Lambda\Omega_\Lambda^*\,. \tag{2.2.1}$$

Moreover, for the corresponding RDMK $F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}$ (see (1.3.3), (1.3.4)) we have that for $\mathbf{x}_0, \mathbf{y}_0 \in \mathcal{C}_a(\Lambda_0)$ with $\sharp\, \mathbf{x}_0 = \sharp\, \mathbf{y}_0$:

$$F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}(\mathbf{x}_0,\mathbf{y}_0) = \int\limits_{\underline{\mathcal{W}}_a^*(\mathbf{x}_0,\mathbf{y}_0)} \chi^{\Lambda_0}(\underline{\Omega}_0^*)\alpha_\Lambda(\underline{\Omega}_0^*)\mathbf{z}^{K(\underline{\Omega}_0^*)}\widehat{q}^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}(\underline{\Omega}_0^*)\underline{\mathbb{P}}^*_{\mathbf{x}_0,\mathbf{y}_0}(\mathrm{d}\underline{\Omega}_0^*). \tag{2.2.2}$$

Here

$$\widehat{q}^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}(\underline{\Omega}_0^*) = \frac{\widehat{\Xi}_\Lambda^{\Lambda_0,\underline{\Omega}_0^*}\big[\Lambda\setminus\Lambda_0|\mathbf{x}(\Lambda^c)\big]}{\Xi\big[\Lambda|\mathbf{x}(\Lambda^c)\big]},\ \underline{\Omega}_0^* \in \underline{\mathcal{W}}_a^*(\mathbf{x}_0,\mathbf{y}_0). \tag{2.2.3}$$

Next, the partition function $\Xi\big[\Lambda|\mathbf{x}(\Lambda^c)\big]$ is defined as in (2.2.1) whereas

$$\begin{aligned}\widehat{\Xi}_\Lambda^{\Lambda_0,\overline{\Omega}_0^*}[\Lambda\setminus\Lambda_0|\mathbf{x}(\Lambda^c)] = &\int\limits_{\mathcal{W}_a^*(\Lambda)} \mathbf{1}\Big(\Omega^*_{\Lambda\setminus\Lambda_0} \in \mathcal{W}_a^*(\Lambda\setminus\Lambda_0)\Big)\chi^{\Lambda_0}(\Omega^*_{\Lambda\setminus\Lambda_0})\alpha_\Lambda(\Omega^*_{\Lambda\setminus\Lambda_0})\\ &\times \frac{\mathbf{z}^{K(\Omega^*_{\Lambda\setminus\Lambda_0})}}{L(\Omega^*_{\Lambda\setminus\Lambda_0})} \exp\Big[-h\Big(\overline{\Omega}_0^* \vee \Omega^*_{\Lambda\setminus\Lambda_0}|\mathbf{x}(\Lambda^c)\Big)\Big]\,\mathrm{d}\Omega^*_{\Lambda\setminus\Lambda_0}.\end{aligned} \tag{2.2.4}$$

Functionals K and L are as in (2.1.11) and (2.1.12). Next, $\chi^{\Lambda_0}(\underline{\Omega}_0^* \vee \Omega^*_{\Lambda\setminus\Lambda_0})$ stands for the indicator requiring that no path $\overline{\omega}^*$ or loop ω^* from the whole collection enters the square Λ_0 at 'control' time points $l\beta$ with $1 \le l < k$, where k equals $k(\overline{\omega}^*)$ or $k(\omega^*)$.

Namely, for a path configuration $\overline{\Omega}_0^* = \{\overline{\omega}^*(1), \dots, \overline{\omega}^*(n)\} \in \overline{\mathcal{W}}_a^*(\mathbf{x}_0,\mathbf{y}_0)$ where $\underline{x} = \{x(1),\dots,x(n)\}$, $\underline{y} = \{y(1),\dots,y(n)\} \in \mathcal{C}(\Lambda_0)$ and a loop configuration $\Omega^*_{\Lambda\setminus\Lambda_0} = \{\omega^*(x), x \in \mathbf{x}_{\Lambda\setminus\Lambda_0}\}$ where $\mathbf{x}_{\Lambda\setminus\Lambda_0} \in \mathcal{C}(\Lambda)$,

$$\begin{aligned}\chi^{\Lambda_0}(\underline{\Omega}_0^* \vee \Omega^*_{\Lambda\setminus\Lambda_0}) = &\,\mathbf{1}\Big(\overline{\omega}^*(j,l\beta) \in \mathbb{R}^d\setminus\Lambda_0\ \forall\, l = 1,\dots,k(\overline{\omega}^*(j))-1, 1 \le j \le n\Big)\\ &\times\mathbf{1}\Big(\omega^*(x,l\beta) \in \mathbb{R}^d\setminus\Lambda_0\ \forall\, l = 1,\dots,k(\omega^*(x))-1, x \in \mathbf{x}(\Lambda)\Big).\end{aligned} \tag{2.2.5}$$

Note that when $k(\overline{\omega}^*(j)) = 1$ or $k(\overline{\omega}^*(x)) = 1$, the above indicator yields no restriction.

Mnemonically, the notation $\widehat{\Xi}^{\Lambda_0,\underline{\Omega}_0^*}$ means the application of an indicator function χ^{Λ_0} in the corresponding integral, together with presence of a specific path configuration $\underline{\Omega}_0^*$ in the energy functional $h\Big(\underline{\Omega}_0^*\vee\Omega^*\big|\mathbf{x}(\Lambda^{\rm c})\Big)$. Pictorially, the quantity $\widehat{\Xi}^{\Lambda_0,\underline{\Omega}_0^*}\big[\Lambda\setminus\Lambda_0|\mathbf{x}(\Lambda^{\rm c})\big]$ in (2.2.3) represents a restricted partition function in $\Lambda\setminus\Lambda_0$ in presence of a path configuration $\underline{\Omega}_0^*$ and in the potential field generated by an external particle configuration $\mathbf{x}(\Lambda^{\rm c})$, with the restriction dictated by χ^{Λ_0}. We would like to note that $\widehat{\Xi}^{\Lambda_0,\underline{\Omega}_0^*}\big[\Lambda\setminus\Lambda_0|\mathbf{x}(\Lambda^{\rm c})\big]$ is only one out of several types of partition functions that we will have to deal with in our analysis.

The aftermath of Lemma 1 is the emergence of a probability measure (PM), $\mu_{\Lambda|\mathbf{x}(\Lambda^{\rm c})}$, on the loop configuration space $\mathcal{W}_a^*(\Lambda)$ (i.e., an RMPP in Λ with marks from the loop space $\mathcal{W}_a^*(0)$). More precisely, $\mu_{\Lambda|\mathbf{x}(\Lambda^{\rm c})}$ is a PM on the standard (Borel) sigma-algebra $\mathfrak{W}^*(\Lambda)$ of subsets of $\mathcal{W}^*(\Lambda)$ supported by $\mathcal{W}_a^*(\Lambda)$.

Definition 4. *The PM $\mu_{\Lambda|\mathbf{x}(\Lambda^{\rm c})}$ is given by the probability density function (PDF) $f_{\Lambda|\mathbf{x}(\Lambda^{\rm c})}(\Omega^*)$, $\Omega^*\in\mathcal{W}_a^*(\Lambda)$, where*

$$f_{\Lambda|\mathbf{x}(\Lambda^{\rm c})}(\Omega^*):=\frac{\mu_{\Lambda|\mathbf{x}(\Lambda^{\rm c})}({\rm d}\Omega^*)}{{\rm d}^{\Lambda}\Omega^*}=\alpha_\Lambda(\Omega^*)\frac{\mathbf{z}^{K(\Omega^*_\Lambda)}}{L(\Omega^*_\Lambda)}\,\frac{\exp\,\Big[-h\Big(\Omega^*_\Lambda\big|\mathbf{x}(\Lambda^{\rm c})\Big)\Big]}{\Xi\big[\Lambda|\mathbf{x}(\Lambda^{\rm c})\big]},\tag{2.2.6}$$

with partition function $\Xi\big[\Lambda|\mathbf{x}(\Lambda^{\rm c})\big]$ as in (1.1.21) and (2.2.1). Furthermore, consider the restriction $\mu^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{\rm c})}$ of $\mu_{\Lambda,\mathbf{x}(\Lambda^{\rm c})}$ to the sigma-algebra $\mathfrak{W}^*(\Lambda_0)$ (more precisely, an induced PM on $(\mathcal{W}^*(\Lambda_0),\mathfrak{W}^*(\Lambda_0))$). Here $\mathfrak{W}^*(\Lambda_0)$ is treated as a sigma-subalgebra of $\mathfrak{W}^*(\Lambda)$, through the map $\mathcal{W}^*(\Lambda)\to\mathcal{W}^*(\Lambda_0)$:

$$\Omega^*=\{\omega^*_x,\,x\in\mathbf{x}(\Lambda)\subset\Lambda\}\mapsto\Omega^*_{\Lambda_0}=\{\omega^*_x:\,x\in\mathbf{x}(\Lambda)\cap\Lambda_0\}.$$

Then, $\mu^{\Lambda_0}_{\Lambda,(\Lambda^{\rm c})}$ is defined by the PDF $f^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{\rm c})}(\Omega^*_0):=\dfrac{\mu^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{\rm c})}({\rm d}\Omega^*_0)}{{\rm d}^{\Lambda_0}\Omega^*_0}$ of the form

$$\begin{aligned}f^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{\rm c})}(\Omega^*_0)&=\mathbf{1}(\Omega^*_0\in\mathcal{W}^*_a(\Lambda_0))\\&\times\alpha_\Lambda(\Omega^*_0)\frac{\mathbf{z}^{K(\Omega^*_0)}}{L(\Omega^*_0)}\,\frac{\Xi^{\Lambda_0,\Omega^*_0}_\Lambda\big[\Lambda\setminus\Lambda_0|\mathbf{x}(\Lambda^{\rm c})\big]}{\Xi\big[\Lambda|\mathbf{x}(\Lambda^{\rm c})\big]},\ \ \Omega^*_0\in\mathcal{W}^*(\Lambda_0).\end{aligned}\tag{2.2.7}$$

Here, the numerator $\Xi^{\Lambda_0,\Omega^*_0}_\Lambda\big[\Lambda\setminus\Lambda_0|\mathbf{x}(\Lambda^{\rm c})\big]$ is given by

$$\begin{aligned}\Xi^{\Lambda_0,\Omega^*_0}_\Lambda\big[\Lambda\setminus\Lambda_0|\mathbf{x}(\Lambda^{\rm c})\big]=&\int\limits_{\mathcal{W}_a(\Lambda)}\mathbf{1}\Big(\Omega^*_{\Lambda\setminus\Lambda_0}\in\mathcal{W}^*_a(\Lambda\setminus\Lambda_0)\Big)\alpha_\Lambda(\Omega^*_{\Lambda\setminus\Lambda_0})\frac{\mathbf{z}^{K(\Omega^*_{\Lambda\setminus\Lambda_0})}}{L(\Omega^*_{\Lambda\setminus\Lambda_0})}\\&\times\exp\,\Big[-h\Big(\Omega^*_0\vee\Omega^*_{\Lambda\setminus\Lambda_0}\big|\mathbf{x}(\Lambda^{\rm c})\Big)\Big]\,{\rm d}^{\Lambda}\Omega^*_{\Lambda\setminus\Lambda_0}.\end{aligned}\tag{2.2.8}$$

Pictorially, the quantity $\Xi^{\Lambda_0,\Omega^*_0}_\Lambda\big[\Lambda\setminus\Lambda_0|\mathbf{x}(\Lambda^{\rm c})\big]$ in (2.2.8) represents a partition function in $\Lambda\setminus\Lambda_0$ in the external field generated by the particle configuration $\mathbf{x}(\Lambda^{\rm c})$, in presence of a loop configuration Ω^*_0 over Λ_0, and with Dirichlet boundary condition in box Λ.

The next assertion, Lemma 2, describes compatibility properties of PMs $\mu_{\Lambda|\mathbf{x}(\Lambda^{\rm c})}$ relative to the choice of an intermediate cube Λ' where $\Lambda_0\subset\Lambda'\subset\Lambda$. This property will allow us to use the same formalism in Section 2.3 when box Λ is replaced with the whole space $\mathbb{R}^d$. The proof of Lemma 2 is a standard (although tedious) manipulation with the Gibbsian form of PM $\mu_{\Lambda|\mathbf{x}(\Lambda^{\rm c})}$ and is omitted.

Lemma 2. *The PM $\mu_{\Lambda|\mathbf{x}(\Lambda^c)}$ satisfies the following property: $\forall\ \Lambda_0 \subset \Lambda$ and $\Omega_0^* \in \mathcal{W}^*(\Lambda_0)$, the PDF $f^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}(\Omega_0^*)$ introduced in (2.2.7) has the form*

$$f^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}(\Omega_0^*) = \mathbf{1}\Big(\Omega_0^* \in \mathcal{W}_a^*(\Lambda_0)\Big)\alpha_\Lambda(\Omega_0^*)\,\frac{\mathbf{z}^{K(\Omega_0^*)}}{L(\Omega_0^*)}q^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}(\Omega_0^*). \tag{2.2.9}$$

Here, for all $\Lambda_0 \subseteq \Lambda' \subseteq \Lambda$, the functional $q^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}(\Omega_0^*)$ admits the representation

$$\begin{aligned} q^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}(\Omega_0^*) = \int\limits_{\mathcal{W}_a^*(\Lambda)} &\mathbf{1}(\Omega^*_{\Lambda\setminus\Lambda'} \in \mathcal{W}_a^*(\Lambda\setminus\Lambda'))\alpha_\Lambda(\Omega^*_{\Lambda\setminus\Lambda'}) \\ &\times \Xi_\Lambda^{\Lambda_0,\Omega_0^*}\Big[\Lambda'\setminus\Lambda_0,\, \Omega^*_{\Lambda\setminus\Lambda'} \vee \mathbf{x}(\Lambda^c)\Big]\ \mathrm{d}\mu_{\Lambda|\mathbf{x}(\Lambda^c)}(\Omega^*_{\Lambda\setminus\Lambda'}). \end{aligned} \tag{2.2.10}$$

Furthermore, for a given $\Omega^*_{\Lambda\setminus\Lambda'} \in \mathcal{W}_a^*(\Lambda\setminus\Lambda')$, the conditional partition function $\Xi_\Lambda^{\Lambda_0,\Omega_0^*}\Big[\Lambda'\setminus\Lambda_0,\, \Omega^*_{\Lambda\setminus\Lambda'} \vee \mathbf{x}(\Lambda^c)\Big]$ is defined in a manner similar to quantity $\Xi^{\Lambda_0,\Omega_0^*}[\Lambda\setminus\Lambda_0|\mathbf{x}(\Lambda^c)]$ in (2.2.8):

$$\begin{aligned} \Xi_\Lambda^{\Lambda_0,\Omega_0^*}\Big[\Lambda'\setminus\Lambda_0,\, \Omega^*_{\Lambda\setminus\Lambda'} \vee \mathbf{x}(\Lambda^c)\Big] = \int\limits_{\mathcal{W}_a^*(\Lambda'\setminus\Lambda_0)} &\alpha_\Lambda\left(\Omega^*_{\Lambda'\setminus\Lambda_0}\right)\frac{\mathbf{z}^{K\left(\Omega^*_{\Lambda'\setminus\Lambda_0}\right)}}{L\left(\Omega^*_{\Lambda'\setminus\Lambda_0}\right)} \\ \times\ \exp\ &\Big[-h\Big(\Omega_0^* \vee \Omega^*_{\Lambda'\setminus\Lambda_0}|\Omega^*_{\Lambda\setminus\Lambda'} \vee \mathbf{x}(\Lambda^c)\Big)\Big]\,\mathrm{d}^\Lambda\Omega^*_{\Lambda'\setminus\Lambda_0}. \end{aligned} \tag{2.2.11}$$

For the RDMK $F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}(\mathbf{x}_0,\mathbf{y}_0)$ we have an integral formula (2.2.2) where the functional $\widehat{q}^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}(\underline{\Omega}_0^*)$ is specified in (2.2.3), (2.2.4). Moreover, the following representation holds: $\forall\ \Lambda_0 \subseteq \Lambda' \subset \Lambda$, $\mathbf{x}_0, \mathbf{y}_0 \in \mathcal{C}_a(\Lambda_0)$ with $\sharp\,\mathbf{x}_0 = \sharp\,\mathbf{y}_0$ and $\underline{\Omega}_0^* \in \underline{\mathcal{W}}_a^*(\mathbf{x}_0,\mathbf{y}_0)$,

$$\begin{aligned} \widehat{q}^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}(\underline{\Omega}_0^*) = \int\limits_{\mathcal{W}_a^*(\Lambda)} &\mathbf{1}(\Omega^*_{\Lambda\setminus\Lambda'} \in \mathcal{W}_a^*(\Lambda\setminus\Lambda'))\chi^{\Lambda_0}\left(\Omega^*_{\Lambda\setminus\Lambda'}\right)\alpha_\Lambda\left(\Omega^*_{\Lambda\setminus\Lambda'}\right) \\ &\times\widehat{\Xi}_\Lambda^{\Lambda_0,\underline{\Omega}_0^*}\Big[\Lambda'\setminus\Lambda_0\big|\,\Omega^*_{\Lambda\setminus\Lambda'} \vee \mathbf{x}(\Lambda^c)\Big]\,\mathrm{d}\mu_{\Lambda|\mathbf{x}(\Lambda^c)}(\Omega^*_{\Lambda\setminus\Lambda'})\,. \end{aligned} \tag{2.2.12}$$

Here, in analogy with (2.2.11), for a given $\Omega^*_{\Lambda\setminus\Lambda'} \in \mathcal{W}_a^*(\Lambda\setminus\Lambda')$,

$$\begin{aligned} \widehat{\Xi}_\Lambda^{\Lambda_0,\underline{\Omega}_0^*}\Big[\Lambda'\setminus\Lambda_0\big|\,\Omega^*_{\Lambda\setminus\Lambda'} \vee \mathbf{x}(\Lambda^c)\Big] = \int\limits_{\mathcal{W}_a^*(\Lambda'\setminus\Lambda_0)} &\chi^{\Lambda_0}\left(\Omega^*_{\Lambda'\setminus\Lambda_0}\right)\alpha_\Lambda\left(\Omega^*_{\Lambda'\setminus\Lambda_0}\right) \\ \times\frac{\mathbf{z}^{K\left(\Omega^*_{\Lambda'\setminus\Lambda_0}\right)}}{L\left(\Omega^*_{\Lambda'\setminus\Lambda_0}\right)}\exp\ &\Big[-h\Big(\overline{\Omega}_0^* \vee \Omega^*_{\Lambda'\setminus\Lambda_0}|\Omega^*_{\Lambda\setminus\Lambda'} \vee \mathbf{x}(\Lambda^c)\Big)\Big]\,\mathrm{d}^\Lambda\Omega^*_{\Lambda'\setminus\Lambda_0}. \end{aligned} \tag{2.2.13}$$

Remark 1. *The presence of term* $\exp\ \Big[-h\Big(\Omega_0^* \vee \Omega^*_{\Lambda'\setminus\Lambda_0}|\Omega^*_{\Lambda\setminus\Lambda'} \vee \mathbf{x}(\Lambda^c)\Big)\Big]$ *in (2.2.11) and* $\exp\ \Big[-h\Big(\underline{\Omega}_0^* \vee \Omega^*_{\Lambda'\setminus\Lambda_0}|\Omega^*_{\Lambda\setminus\Lambda'} \vee \mathbf{x}(\Lambda^c)\Big)\Big]$ *in (2.2.14) implies the presence of the related indicators* $\mathbf{1}\Big(\Omega_0^* \vee \Omega^*_{\Lambda'\setminus\Lambda_0} \vee \Omega^*_{\Lambda\setminus\Lambda'} \in \mathcal{W}_a^*(\Lambda)\Big)$ *and* $\mathbf{1}\Big((\underline{\Omega}_0^*, \Omega^*_{\Lambda'\setminus\Lambda_0} \vee \Omega^*_{\Lambda\setminus\Lambda'}) \in \underline{\mathcal{W}}_a^*(\Lambda_0, \Lambda\setminus\Lambda_0)\Big)$.

In particular, for $\Lambda_0 = \Lambda'$, Equations (2.2.10) and (2.2.12) take the form:

$$\begin{aligned} q^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}(\Omega_0^*) = \int\limits_{\mathcal{W}_a^*(\Lambda)} &\mathbf{1}\left(\Omega^*_{\Lambda\setminus\Lambda_0} \in \mathcal{W}_a^*(\Lambda\setminus\Lambda_0)\right)\alpha_\Lambda\left(\Omega^*_{\Lambda\setminus\Lambda_0}\right) \\ &\times\exp\ \Big[-h\Big(\Omega_0^*|\Omega^*_{\Lambda\setminus\Lambda_0} \vee \mathbf{x}(\Lambda^c)\Big)\Big]\,\mathrm{d}\mu_{\Lambda|\mathbf{x}(\Lambda^c)}(\Omega^*_{\Lambda\setminus\Lambda_0}) \end{aligned} \tag{2.2.14}$$

and

$$\begin{aligned}\widehat{q}^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}(\underline{\Omega}^*_0) = \int\limits_{\mathcal{W}^*_a(\Lambda)} &\mathbf{1}\left(\Omega^*_{\Lambda\setminus\Lambda_0} \in \mathcal{W}^*_a(\Lambda\setminus\Lambda_0)\right)\chi^{\Lambda_0}\left(\Omega^*_{\Lambda\setminus\Lambda_0}\right)\alpha_\Lambda\left(\Omega^*_{\Lambda\setminus\Lambda_0}\right)\\ &\times\exp\left[-h\left(\underline{\Omega}^*_0|\Omega^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{x}(\Lambda^c)\right)\right]\mathrm{d}\mu_{\Lambda|\mathbf{x}(\Lambda^c)}(\Omega^*_{\Lambda\setminus\Lambda_0}).\end{aligned} \tag{2.2.15}$$

On the other hand, when $\Lambda' = \Lambda$, Equation (2.2.12) coincides with Equation (2.2.4).

We would like to stress here that the integral $\int \mathrm{d}\mu_{\Lambda|\mathbf{x}(\Lambda^c)}(\Omega^*_{\Lambda\setminus\Lambda'})$ in (2.2.10) and (2.2.12) is taken in the variable $\Omega^*_{\Lambda\setminus\Lambda'}$ considered as an element of space $\mathcal{W}^*_a(\Lambda)$. Likewise, $\int \mathrm{d}\mu_{\Lambda|\mathbf{x}(\Lambda^c)}(\Omega^*_{\Lambda\setminus\Lambda'})$ in (2.2.14) and (2.2.15) is taken in $\Omega^*_{\Lambda\setminus\Lambda_0}$ also considered as an element of space $\mathcal{W}^*_a(\Lambda)$. This explains the absence in (2.2.10), (2.2.12), (2.2.14) and (2.2.15) of the denominator $\Xi\left[\Lambda|\mathbf{x}(\Lambda^c)\right]$ figuring in (2.2.4), (2.2.6) and (2.2.7). On the other hand, integration $\int \mathrm{d}^\Lambda\Omega^*_{\Lambda'\setminus\Lambda_0}$ in (2.2.11) and (2.2.13) is in variable $\Omega^*_{\Lambda'\setminus\Lambda_0} \in \mathcal{W}^*_a(\Lambda'\setminus\Lambda_0)$.

Equations (2.2.9)–(2.2.13) are called the *FK-DLR equations* in volume Λ.

An important property is given in Lemma 3 establishing uniform estimates for quantities $\widehat{q}^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}(\underline{\Omega}^*_0)$ (cf. Equations (2.2.3) and (2.2.12)) and $q^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}(\Omega^*_0)$ (see Equations (2.2.10) and (2.2.14)). Recall that W^- has been specified in (1.1.3a).

Lemma 3. *The following bounds are satisfied:*

(a) for a path collection $\underline{\Omega}^*_0 = (\overline{\omega}^*(1),\ldots,\overline{\omega}^*(n_0)) \in \underline{\mathcal{W}}^*_a(\mathbf{x}_0,\mathbf{y}_0)$ with $\sharp\,\mathbf{x}_0 = \sharp\,\mathbf{y}_0 = n_0$,

$$\begin{aligned}\widehat{q}^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}(\underline{\Omega}^*_0) &= \frac{1}{\Xi\left[\Lambda|\mathbf{x}(\Lambda^c)\right]}\int\limits_{\mathcal{W}^*_a(\Lambda\setminus\Lambda_0)}\chi^{\Lambda_0}(\Omega^*_{\Lambda\setminus\Lambda_0})\alpha_\Lambda(\Omega^*_{\Lambda\setminus\Lambda_0})\\ &\times\frac{\mathbf{z}^{K(\Omega^*_{\Lambda\setminus\Lambda_0})}}{L(\Omega^*_{\Lambda\setminus\Lambda_0})}\mathrm{e}^{-h(\underline{\Omega}^*_0\vee\Omega^*_{\Lambda\setminus\Lambda_0}|\mathbf{x}(\Lambda^c))}\mathrm{d}\Omega^*_{\Lambda\setminus\Lambda_0} \le \exp\left[\beta K(\underline{\Omega}^*_0)\mathrm{W}^-\right].\end{aligned} \tag{2.2.16}$$

(b) for a loop configuration $\Omega^*_0 \in \mathcal{W}^*_a(\Lambda_0)$,

$$\begin{aligned}q^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}(\Omega^*_0) &= \frac{1}{\Xi\left[\Lambda|\mathbf{x}(\Lambda^c)\right]}\int\limits_{\mathcal{W}^*_a(\Lambda\setminus\Lambda_0)}\chi^{\Lambda_0}(\Omega^*_{\Lambda\setminus\Lambda_0})\alpha_\Lambda(\Omega^*_{\Lambda\setminus\Lambda_0})\\ &\times\frac{\mathbf{z}^{K(\Omega^*_{\Lambda\setminus\Lambda_0})}}{L(\Omega^*_{\Lambda\setminus\Lambda_0})}\mathrm{e}^{-h(\Omega^*_0\vee\Omega^*_{\Lambda\setminus\Lambda_0}|\mathbf{x}(\Lambda^c))}\mathrm{d}\Omega^*_{\Lambda\setminus\Lambda_0} \le \exp\left[\beta K(\Omega^*_0)\mathrm{W}^-\right].\end{aligned} \tag{2.2.17}$$

Proof. Both inequalities (2.2.16) and (2.2.17) are demonstrated by using similar arguments. Thus, we focus on one of them, say (2.2.16), and analyse Equations (2.1.1)–(2.1.11). Observe that the integral for $h(\underline{\Omega}^*\vee\Omega^*|\mathbf{x}(\Lambda^c))$ in (2.1.8)–(2.1.10) comprises contributions $h(\overline{\omega}^*(j))$, $h(\overline{\omega}^*(j),\overline{\omega}^*(j'))$, $h(\overline{\omega}^*(j)||\Omega^*)$ and $h(\overline{\omega}^*(j)||\mathbf{x}(\Lambda^c))$ from paths $\overline{\omega}^*(j)$, $\overline{\omega}^*(j')$ (cf. (2.1.1), (2.1.3), (2.1.7) and (2.1.10)). In terms of integrals $\int\limits_0^\beta \mathrm{dt}$, we have to lower-bound the classical energy of interaction between a single particle and a particle configuration (possibly infinite) in $\mathbb{R}^d$. According to the definition of the value W^-, we obtain that $\forall\,\Omega^*_{\Lambda\setminus\Lambda_0} \in \mathcal{W}^*_a(\Lambda\setminus\Lambda_0)$,

$$\mathrm{e}^{-h(\underline{\Omega}^*_0\vee\Omega^*_{\Lambda\setminus\Lambda_0}|\mathbf{x}(\Lambda^c))} \le \mathrm{e}^{\beta K(\underline{\Omega}^*_0)\mathrm{W}^-}\times\mathrm{e}^{-h(\Omega^*_{\Lambda\setminus\Lambda_0}|\mathbf{x}(\Lambda^c))}. \tag{2.2.18}$$

This yields that $\widehat{q}^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}(\underline{\Omega}^*_0) \leq \mathrm{e}^{\beta K(\Omega^*_0)W^-}\dfrac{\Xi_\Lambda\left[\Lambda\setminus\Lambda_0|\mathbf{x}(\Lambda^c)\right]}{\Xi\left[\Lambda|\mathbf{x}(\Lambda^c)\right]}$, where

$$\Xi_\Lambda\left[\Lambda\setminus\Lambda_0|\mathbf{x}(\Lambda^c)\right] = \int\limits_{\mathcal{W}^*_a(\Lambda\setminus\Lambda_0)} \chi^{\Lambda_0}(\Omega^*_{\Lambda\setminus\Lambda_0})\alpha_\Lambda(\Omega^*_{\Lambda\setminus\Lambda_0})\frac{\mathbf{z}^{K(\Omega^*_{\Lambda\setminus\Lambda_0})}}{L(\Omega^*_{\Lambda\setminus\Lambda_0})}\mathrm{e}^{-h(\Omega^*_{\Lambda\setminus\Lambda_0}|\mathbf{x}(\Lambda^c))}\mathrm{d}\Omega^*_{\Lambda\setminus\Lambda_0}.$$

Since $\Xi_\Lambda\left[\Lambda\setminus\Lambda_0|\mathbf{x}(\Lambda^c)\right] \leq \Xi\left[\Lambda|\mathbf{x}(\Lambda^c)\right]$, inequality (2.2.16) follows. $\square$

2.3. The Infinite-Volume FK-DLR Equations and RDMKs

The infinite-volume versions of the RDMK arise when we mimic properties listed in Lemmas 1 and 2 by getting rid of the reference to the enveloping box Λ (including the external particle configuration $\mathbf{x}(\Lambda^c)$ and the functional α_Λ indicating Dirichlet's boundary condition). The first place to do so is the PM $\mu_{\Lambda|\mathbf{x}(\Lambda^c)}$; to this end we need to consider its infinite-volume analog $\mu_{\mathbb{R}^d}$ representing an RMPP in the whole space $\mathbb{R}^d$. Formally, $\mu_{\mathbb{R}^d}$ yields a PM on the sigma-algebra $\mathfrak{W}^*\left(\mathbb{R}^d\right)$ of subsets in $\mathcal{W}^*\left(\mathbb{R}^d\right)$. The space $\mathcal{W}^*\left(\mathbb{R}^d\right)$ is formed by pairs $\left[\mathbf{x}(\mathbb{R}^d), \mathbf{\Omega}^*(\mathbf{x}(\mathbb{R}^d))\right]$ where $\mathbf{x}(\mathbb{R}^d)$ is a locally finite set in the plane, and $\mathbf{\Omega}^*(\mathbf{x}(\mathbb{R}^d))$ (in brief, $\mathbf{\Omega}^*_{\mathbb{R}^d}$ or simply $\mathbf{\Omega}^*$) is a collection $\{\omega^*(x),\ x\in\mathbf{x}(\mathbb{R}^d)\}$ of loops $\omega^*(x)\in\mathcal{W}^*_x$. Alternatively, $\mathbf{\Omega}^*(\mathbf{x}(\mathbb{R}^d)) \in \mathop{\times}\limits_{x\in\mathbf{x}(\Lambda)}\mathcal{W}^*_x$. Next, $\mathfrak{W}^*\left(\mathbb{R}^d\right)$ is the sigma-algebra of subsets in $\mathcal{W}^*\left(\mathbb{R}^d\right)$ generated by the cylinder events.

To simplify technical aspects of the presentation, we will omit the reference to the initial configuration $\mathbf{x}(\mathbb{R}^d)$ and write $\mathbf{\Omega}_{\mathbb{R}^d}\in\mathcal{W}^*(\mathbb{R}^d)$ or $\mathbf{\Omega}^*\in\mathcal{W}^*(\mathbb{R}^d)$ (given a loop configuration $\mathbf{\Omega}^*$, the initial particle configuration is uniquely determined and is denoted by $\mathbf{x}(\mathbf{\Omega}^*)$).

Furthermore, we will use the notation $\mathcal{W}^*(\Lambda^c)$ for the subset in $\mathcal{W}^*(\mathbb{R}^d)$ formed by loop configurations $\mathbf{\Omega}^*_{\Lambda^c}$ with $\mathbf{x}(\mathbf{\Omega}^*_{\Lambda^c})\in\mathcal{C}_a(\Lambda^c)$. (We call such $\mathbf{\Omega}^*_{\Lambda^c}$ a loop configuration over Λ^c.)

Definition 5. *We say that a PM $\mu=\mu_{\mathbb{R}^d}$ on $\left(\mathcal{W}^*\left(\mathbb{R}^d\right), \mathfrak{W}^*\left(\mathbb{R}^d\right)\right)$ satisfies the (infinite-volume) FK-DLR equations if the restriction μ^{Λ_0} of μ to $\mathfrak{W}^*(\Lambda_0)$ is given by the PDF $f^{\Lambda_0}(\Omega^*_0) := \dfrac{\mu^{\Lambda_0}(\mathrm{d}\Omega^*_0)}{\mathrm{d}^{\Lambda_0}\Omega^*_0}$ of the form*

$$f^{\Lambda_0}(\Omega^*_0) = \mathbf{1}\left(\Omega^*_0\in\mathcal{W}^*_a(\Lambda_0)\right)\frac{\mathbf{z}^{K(\Omega^*_0)}}{L(\Omega^*_0)}q^{\Lambda_0}(\Omega^*_0),\ \ \Omega^*_0\in\mathcal{W}^*(\Lambda_0). \tag{2.3.1}$$

Here, the functional $q^{\Lambda_0}(\Omega^*_0)$ admits the following representation: $\forall$ pair of cubes $\Lambda_0\subset\Lambda\subset\mathbb{R}^d$,

$$q^{\Lambda_0}(\Omega^*_0) = \int_{\mathcal{W}^*_a(\mathbb{R}^d)}\mathbf{1}\left(\mathbf{\Omega}^*_{\Lambda^c}\in\mathcal{W}^*_a(\Lambda^c)\right)\Xi^{\Lambda_0,\Omega^*_0}\left(\Lambda\setminus\Lambda_0\middle|\mathbf{\Omega}^*_{\Lambda^c}\right)\mathrm{d}\mu(\mathbf{\Omega}^*_{\Lambda^c}). \tag{2.3.2}$$

Observe similarities with Equation (2.2.9). At the same time, note the absence the indicator α_Λ in the RHS of (2.3.2). Here, for a given (infinite) loop configuration $\mathbf{\Omega}^*_{\Lambda^c}\in\mathcal{W}^*_a(\Lambda^c)$, the expression $\Xi^{\Lambda_0,\Omega^*_0}\left(\Lambda\setminus\Lambda_0\middle|\mathbf{\Omega}^*_{\Lambda^c}\right)$ yields a partition function in $\Lambda\setminus\Lambda_0$, in the external field generated by $\mathbf{\Omega}^*_{\Lambda^c}$ and in presence of a loop configuration $\Omega^*_0\in\mathcal{W}^*_a(\Lambda_0)$ (but without a boundary conditions):

$$\begin{aligned}\Xi^{\Lambda_0,\Omega^*_0}\left(\Lambda\setminus\Lambda_0\middle|\mathbf{\Omega}^*_{\Lambda^c}\right) = &\int\limits_{\mathcal{W}^*_a(\Lambda\setminus\Lambda_0)}\frac{\mathbf{z}^{K(\Omega^*_{\Lambda\setminus\Lambda_0})}}{L(\Omega^*_{\Lambda\setminus\Lambda_0})}\mathbf{1}\left(\Omega^*_0\vee\Omega^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{\Omega}^*_{\Lambda^c}\in\mathcal{W}^*_a(\mathbb{R}^d)\right)\\ &\times\exp\left[-h\left(\Omega^*_0\vee\Omega^*_{\Lambda\setminus\Lambda_0}\middle|\mathbf{\Omega}^*_{\Lambda^c}\right)\right]\mathrm{d}\Omega^*_{\Lambda\setminus\Lambda_0}.\end{aligned} \tag{2.3.3}$$

$$\Big(\Xi^{\Lambda_0,\Omega_0^*}\left[\Lambda'\setminus\Lambda_0,\,\Omega^*_{\Lambda\setminus\Lambda'}\vee\mathbf{x}(\Lambda^c)\right]=\int\limits_{\mathcal{W}_a^*(\Lambda'\setminus\Lambda_0)}\alpha_\Lambda\left(\Omega^*_{\Lambda'\setminus\Lambda_0}\right)\frac{\mathrm{z}^{K\left(\Omega^*_{\Lambda'\setminus\Lambda_0}\right)}}{L\left(\Omega^*_{\Lambda'\setminus\Lambda_0}\right)}$$
$$\times\exp\left[-h\Big(\Omega_0^*\vee\Omega^*_{\Lambda'\setminus\Lambda_0}\big|\Omega^*_{\Lambda\setminus\Lambda'}\vee\mathbf{x}(\Lambda^c)\Big)\right]\mathrm{d}^\Lambda\Omega^*_{\Lambda'\setminus\Lambda_0}.\Big) \tag{2.2.11}$$

Comparing to Equation (2.2.11), we see a difference: the integral $\int_{\mathcal{W}_a^*(\Lambda\setminus\Lambda_0)}\mathrm{d}\Omega^*_{\Lambda\setminus\Lambda_0}$ *in (2.3.3) provides a simplification. In turn,* $h\left(\Omega_0^*\vee\Omega^*_{\Lambda\setminus\Lambda_0}|\boldsymbol{\Omega}^*_{\Lambda^c}\right)$ *represents the energy of the concatenated loop configuration* $\Omega_0^*\vee\Omega^*_{\Lambda\setminus\Lambda_0}$ *over* Λ*, in the external potential generated by the loop configuration* $\boldsymbol{\Omega}^*_{\Lambda^c}$ *over* Λ^c*. Formally,* $h\left(\Omega_0^*\vee\Omega^*_{\Lambda\setminus\Lambda_0}|\boldsymbol{\Omega}^*_{\Lambda^c}\right)$ *is defined, for* $\Omega_0^*\vee\Omega^*_{\Lambda\setminus\Lambda_0}\vee\boldsymbol{\Omega}^*_{\Lambda^c}\in\mathcal{W}_a^*(\mathbb{R}^d)$*, as the limit:*

$$h\left(\Omega_0^*\vee\Omega^*_{\Lambda\setminus\Lambda_0}|\boldsymbol{\Omega}^*_{\Lambda^c}\right)=\lim_{\widetilde{\Lambda}\nearrow\mathbb{R}^d}h\left[\Omega_0^*\vee\Omega^*_{\Lambda\setminus\Lambda_0}|\boldsymbol{\Omega}^*_{\widetilde{\Lambda}\setminus\Lambda}\right]. \tag{2.3.4}$$

Here, $\widetilde{\Lambda}$ *stands for the cube* $[-\widetilde{\mathrm{L}},\widetilde{\mathrm{L}}]^d$ *of side-length* $2\widetilde{\mathrm{L}}$ *centered at the origin in* $\mathbb{R}^d$*, and* $\boldsymbol{\Omega}^*_{\widetilde{\Lambda}\setminus\Lambda}$ *denotes the restriction of* $\boldsymbol{\Omega}^*_{\Lambda^c}$ *to* $\widetilde{\Lambda}\setminus\Lambda$*. The limit* $\lim_{\widetilde{\Lambda}\nearrow\mathbb{R}^d}$ *in (2.3.4) means that* $\widetilde{\mathrm{L}}\to\infty$*. The Equations (2.3.1)–(2.3.4) are referred to as infinite-volume FK-DLR equations.*

For short, a measure $\mu=\mu_{\mathbb{R}^d}$ *satisfying (2.3.1)–(2.3.4) is called an FK-DLR probability measure (FK-DLR PM). The class of FK-DLR PMs (for a given pair of values* $\mathrm{z}\in(0,1),\beta\in(0,+\infty)$*) is denoted by* $\mathfrak{K}(\mathrm{z},\beta)$*, or, briefly,* $\mathfrak{K}$*. It is straightforward that any PM* $\mu\in\mathfrak{K}$ *is supported by the set* $\mathcal{W}_a^*(\mathbb{R}^d)$*:* $\mu(\mathcal{W}_a^*(\mathbb{R}^2))=1$*.*

Definition 6. *Let* $\mu\in\mathfrak{K}(\mathrm{z},\beta)$ *be an FK-DLR PM. In this definition we associate with* μ *a family of integral kernels* $F^{\Lambda_0}(\mathbf{x}_0,\mathbf{y}_0)$ *where* $\mathbf{x}_0,\mathbf{y}_0\in\mathcal{C}(\Lambda_0)$ *and* $\Lambda_0=\underset{1\le j\le d}{\times}[\mathrm{b}^j-\mathrm{L}_0,\mathrm{b}^j+\mathrm{L}_0]\subset\mathbb{R}^d$ *is an arbitrary cube. Namely, when* $\sharp\,\mathbf{x}_0=\sharp\,\mathbf{y}_0$*, we set, similarly to (2.2.2):*

$$F^{\Lambda_0}(\mathbf{x}_0,\mathbf{y}_0)=\int_{\underline{\mathcal{W}}_a^*(\mathbf{x}_0,\mathbf{y}_0)}\mathrm{z}^{K(\underline{\Omega}_0^*)}\chi^{\Lambda_0}(\underline{\Omega}_0^*)\widehat{q}^{\Lambda_0}(\underline{\Omega}_0^*)\mathrm{d}\underline{\mathbb{P}}^*_{\mathbf{x}_0,\mathbf{y}_0}(\underline{\Omega}_0^*). \tag{2.3.5}$$

In turn, the quantity $\widehat{q}^{\Lambda_0}(\underline{\Omega}_0^*)$ *admits the following integral representation involving PM* μ*:* $\forall$ *cube* $\Lambda'\subset\mathbb{R}^d$ *containing* Λ_0*,*

$$\widehat{q}^{\Lambda_0}(\underline{\Omega}_0^*)=\int\limits_{\mathcal{W}_a^*(\mathbb{R}^d)}\mathbf{1}\left(\boldsymbol{\Omega}^*_{\mathbb{R}^d\setminus\Lambda'}\in\mathcal{W}_a^*(\mathbb{R}^d\setminus\Lambda')\right)\chi^{\Lambda_0}(\boldsymbol{\Omega}^*_{\mathbb{R}^d\setminus\Lambda'})$$
$$\times\,\widehat{\Xi}^{\Lambda_0,\underline{\Omega}_0^*}\left[\Lambda'\setminus\Lambda_0|\boldsymbol{\Omega}^*_{\mathbb{R}^d\setminus\Lambda'}\right]\mathrm{d}\mu(\boldsymbol{\Omega}^*_{\mathbb{R}^d\setminus\Lambda'}). \tag{2.3.6}$$

In particular, for $\Lambda'=\Lambda_0$*:*

$$\widehat{q}^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}(\underline{\Omega}_0^*)=\int\limits_{\mathcal{W}_a^*(\mathbb{R}^d)}\mathbf{1}\left(\boldsymbol{\Omega}^*_{\mathbb{R}^d\setminus\Lambda_0}\in\mathcal{W}_a^*(\mathbb{R}^d\setminus\Lambda_0)\right)\chi^{\Lambda_0}(\boldsymbol{\Omega}^*_{\mathbb{R}^d\setminus\Lambda_0})$$
$$\times\exp\left[-h\left(\underline{\Omega}_0^*|\boldsymbol{\Omega}^*_{\mathbb{R}^d\setminus\Lambda_0}\right)\right]\mathrm{d}\mu(\boldsymbol{\Omega}^*_{\mathbb{R}^d\setminus\Lambda_0}). \tag{2.3.7}$$

Again note similarities and differences with (2.2.13) and (2.2.16). For instance, the partition function $\widehat{\Xi}^{\Lambda_0,\underline{\Omega}_0^*}\left[\Lambda'\setminus\Lambda_0|\boldsymbol{\Omega}^*_{\mathbb{R}^d\setminus\Lambda'}\right]$ *in (2.3.6) has the form analogous to* $\widehat{\Xi}_\Lambda^{\Lambda_0,\underline{\Omega}_0^*}\left(\Lambda'\setminus\Lambda_0|\Omega^*_{\Lambda\setminus\Lambda'}\vee\mathbf{x}(\Lambda^c)\right)$ *in (2.2.14):*

$$\widehat{\Xi}^{\Lambda_0,\underline{\Omega}_0^*}\left[\Lambda'\setminus\Lambda_0|\boldsymbol{\Omega}^*_{\mathbb{R}^d\setminus\Lambda'}\right]=\int\limits_{\mathcal{W}_a^*(\Lambda'\setminus\Lambda_0)}\chi^{\Lambda_0}\left(\Omega^*_{\Lambda'\setminus\Lambda_0}\right)\frac{\mathrm{z}^{K\left(\Omega^*_{\Lambda'\setminus\Lambda_0}\right)}}{L\left(\Omega^*_{\Lambda'\setminus\Lambda_0}\right)}$$
$$\times\exp\left[-h\left(\underline{\Omega}_0^*\vee\Omega^*_{\Lambda'\setminus\Lambda_0}|\boldsymbol{\Omega}^*_{\mathbb{R}^d\setminus\Lambda'}\right)\right]\mathrm{d}\Omega^*_{\Lambda'\setminus\Lambda_0}. \tag{2.3.8}$$

The indicators $\chi^{\Lambda_0}(\underline{\Omega}_0^*)$, $\chi^{\Lambda_0}(\overline{\mathbf{\Omega}}^*_{\mathbb{R}^d\setminus\Lambda'})$ and $\chi^{\Lambda_0}(\Omega^*_{\Lambda'\setminus\Lambda_0})$ in (2.3.5)–(2.3.8) are defined similarly to (2.2.5). Finally, similarly to (2.3.4),

$$h\left(\underline{\Omega}_0^* \vee \Omega^*_{\Lambda'\setminus\Lambda_0}|\mathbf{\Omega}^*_{\mathbb{R}^d\setminus\Lambda'}\right) = \lim_{\widetilde{\Lambda}\nearrow\mathbb{R}^d} h\left[\underline{\Omega}_0^* \vee \Omega^*_{\Lambda\setminus\Lambda_0}|\mathbf{\Omega}^*_{\widetilde{\Lambda}\setminus\Lambda'}\right]. \tag{2.3.9}$$

When $\sharp\,\mathbf{x}_0 \neq \sharp\,\mathbf{y}_0$, we set: $F^{\Lambda_0}(\mathbf{x}_0,\mathbf{y}_0) = 0$.

It is instructive to re-write the definitions (2.3.4) and (2.3.9) in line with (2.1.2), (2.1.4), (2.1.8) and (2.1.10), expressing the functionals $h\left(\Omega_0^* \vee \mathbf{\Omega}^*_{\Lambda\setminus\Lambda_0}|\mathbf{\Omega}^*_{\Lambda^c}\right)$ and $h\left(\underline{\Omega}_0^* \vee \Omega^*_{\Lambda\setminus\Lambda_0}|\mathbf{\Omega}^*_{\Lambda^c}\right)$ in terms of energies of particle configurations $\Omega_0^*(\mathtt{t}) \vee \Omega^*_{\Lambda\setminus\Lambda_0}(\mathtt{t})$, $\underline{\Omega}_0^*(\mathtt{t}) \vee \Omega^*_{\Lambda\setminus\Lambda_0}(\mathtt{t})$ and $\mathbf{\Omega}^*_{\Lambda^c}(\mathtt{t})$ forming $\mathtt{t}$-sections of the corresponding loop and path collections, where $0 \leq \mathtt{t} \leq \beta$. Namely,

$$h\left(\Omega_0^* \vee \Omega^*_{\Lambda\setminus\Lambda_0}|\mathbf{\Omega}^*_{\Lambda^c}\right) = \int_0^\beta E\left[\Omega_0^*(\mathtt{t}) \vee \Omega^*_{\Lambda\setminus\Lambda_0}(\mathtt{t})|\mathbf{\Omega}^*_{\Lambda^c}(\mathtt{t})\right]\mathrm{d}\mathtt{t} \tag{2.3.10}$$

where

$$\begin{aligned} &E\left[\Omega_0^*(\mathtt{t}) \vee \Omega^*_{\Lambda\setminus\Lambda_0}(\mathtt{t})|\mathbf{\Omega}^*_{\Lambda^c}(\mathtt{t})\right] \\ &\quad = E\left[\Omega_0^*(\mathtt{t}) \vee \Omega^*_{\Lambda\setminus\Lambda_0}(\mathtt{t})\right] + E\left[\underline{\Omega}_0^*(\mathtt{t}) \vee \Omega^*_{\Lambda\setminus\Lambda_0}(\mathtt{t})||\mathbf{\Omega}^*_{\Lambda^c}(\mathtt{t})\right] \end{aligned} \tag{2.3.11}$$

and

$$\begin{aligned} &E\left[\underline{\Omega}_0^*(\mathtt{t}) \vee \Omega^*_{\Lambda\setminus\Lambda_0}(\mathtt{t})|\mathbf{\Omega}^*_{\Lambda^c}(\mathtt{t})\right] \\ &\quad = E\left[\underline{\Omega}_0^*(\mathtt{t}) \vee \Omega^*_{\Lambda\setminus\Lambda_0}(\mathtt{t})\right] + E\left[\underline{\Omega}_0^*(\mathtt{t}) \vee \Omega^*_{\Lambda\setminus\Lambda_0}(\mathtt{t})||\mathbf{\Omega}^*_{\Lambda^c}(\mathtt{t})\right]. \end{aligned} \tag{2.3.12}$$

In turn, finite particle configurations $\Omega_0^*(\mathtt{t})$, $\underline{\Omega}_0^*(\mathtt{t})$ and $\Omega^*_{\Lambda\setminus\Lambda_0}(\mathtt{t})$ and an infinite particle configuration $\mathbf{\Omega}^*_{\Lambda^c}(\mathtt{t})$ are given by

$$\Omega_0^*(\mathtt{t}) = \bigcup_{\omega^*\in\Omega_0^*}\ \bigcup_{0\leq l<k(\omega^*)}\{\omega^*(l\beta+\mathtt{t})\},\ \underline{\Omega}_0^*(\mathtt{t}) = \bigcup_{\overline{\omega}^*\in\underline{\Omega}_0^*}\ \bigcup_{0\leq l<k(\overline{\omega}^*)}\{\overline{\omega}^*(l\beta+\mathtt{t})\},$$

$$\Omega^*_{\Lambda\setminus\Lambda_0}(\mathtt{t}) = \bigcup_{\omega^*\in\Omega^*_{\Lambda\setminus\Lambda_0}}\ \bigcup_{0\leq l<k(\omega^*)}\{\omega^*(l\beta+\mathtt{t})\}$$

and

$$\begin{aligned} \mathbf{\Omega}^*_{\Lambda^c}(\mathtt{t}) &= \bigcup_{\omega^*\,\in\,\mathbf{\Omega}^*_{\Lambda^c}}\ \bigcup_{0\leq l<k(\overline{\omega}^*)}\{\omega^*(l\beta+\mathtt{t})\} \\ &= \left\{\omega^*(l\beta+\mathtt{t}):\ \omega^*\in\mathbf{\Omega}^*_{\Lambda^c},\ 0\leq l<k(\overline{\omega}^*)\right\}. \end{aligned}$$

Owing to the FK-DLR property of μ, the RHS in (2.3.6) does not depend on the choice of cube $\Lambda \supset \Lambda_0$. Moreover, the kernels F^{Λ_0} satisfy the compatibility property: $\forall$ pair of cubes $\Lambda_1 \subset \Lambda_0$,

$$\int_{\mathcal{C}(\Lambda_0\setminus\Lambda^1)} F^{\Lambda_0}(\mathbf{x}_1 \vee \mathbf{z}, \mathbf{y}_1 \vee \mathbf{z})\mathrm{d}\mathbf{z} = F^{\Lambda_1}(\mathbf{x}_1,\mathbf{y}_1),\ \ \mathbf{x}_1,\mathbf{y}_1 \in \mathcal{C}(\Lambda_0). \tag{2.3.13}$$

In particular,

$$\int_{\mathcal{C}(\Lambda_0)} F^{\Lambda_0}(\mathbf{z},\mathbf{z})\mathrm{d}\mathbf{z} = 1. \tag{2.3.14}$$

Definition 7. *Let $\mu \in \mathfrak{K}(z,\beta)$ be FK-DLR and $\{F^{\Lambda_0}\}$ be the family of kernels associated with μ by Equations (2.3.4)–(2.3.8). Given a cube Λ_0, introduce a trace-class operator R^{Λ_0} acting on $\phi_{\Lambda_0} \in \mathcal{H}(\Lambda_0)$:*

$$R^{\Lambda_0}\phi_{\Lambda_0}(\mathbf{x}_0) = \int_{\mathcal{C}(\Lambda_0)} F^{\Lambda_0}(\mathbf{x}_0,\mathbf{y}_0)\phi_{\Lambda_0}(\mathbf{y}_0)\mathrm{d}\mathbf{y}_0,\ \ \mathbf{x}_0 \in \mathcal{C}(\Lambda_0). \tag{2.3.15}$$

Then, according to (2.3.13)–(2.3.14), for $\Lambda_1 \subset \Lambda_0$,

$$\mathrm{tr}_{\mathcal{H}(\Lambda_0\setminus\Lambda_1)} R^{\Lambda_0} = R^{\Lambda_1}, \ \ \mathrm{tr}_{\mathcal{H}(\Lambda_0)} R^{\Lambda_0} = 1. \tag{2.3.16}$$

The family of operators R^{Λ_0} defines a linear normalized functional on the quasilocal C^*-algebra $\mathfrak{B}(\mathbb{R}^d)$ such that for $A \in \mathfrak{B}(\Lambda_0)$

$$\varphi(A) = \mathrm{tr}_{\mathcal{H}(\Lambda_0)}\big(AR^{\Lambda_0}\big). \tag{2.3.17}$$

We call the functional $A \in \mathfrak{B}(\mathbb{R}^d) \mapsto \varphi(A)$ an FK-DLR functional generated by μ; to stress this fact, we sometimes use the notation φ_μ. If in addition φ is a state (that is, the operators R^{Λ_0} are positive-definite), then we say that φ is an FK-DLR state. In this case, we call the operator R^{Λ_0} an infinite-volume FK-DLR RDM. The class of FK-DLR functionals is denoted by $\mathfrak{F} = \mathfrak{F}(\mathrm{z}, \beta)$ and its subset consisting of the FK-DLR states by $\mathfrak{F}_+ = \mathfrak{F}_+(\mathrm{z}, \beta)$.

Before we move further, we would like to introduce a property conventionally called a 'Ruelle superstability bound'. It is closely related to the so-called Campbell formula assessing integrals of summatory functions $\Sigma_g : \mathbf{\Omega}^* \in \mathcal{W}^*_a(\mathbb{R}^d) \mapsto \sum\limits_{\Omega^*\subset\mathbf{\Omega}^*} g(\Omega^*)$:

$$\int_{\mathcal{W}^*_a(\mathbb{R}^d)} \Sigma_g(\mathbf{\Omega}^*)\mu(\mathrm{d}\mathbf{\Omega}^*) = \int_{\mathcal{W}^{*,\mathrm{f}}_a(\mathbb{R}^d)} \mathrm{M}(\Omega^*)g(\Omega^*)\mathrm{d}\Omega^*. \tag{2.3.18}$$

Equation (2.3.18) is considered for a given RMPP ν (i.e., for a PM ν on loop configuration space $(\mathcal{W}^*(\mathbb{R}^d), \mathfrak{W}^*(\mathbb{R}^d))$) and all test-functions $\Omega^* \in \mathcal{W}^{*,\mathrm{f}}_a(\mathbb{R}^d) \mapsto g(\Omega^*) \geq 0$; it determines a moment function $\Omega^* \in \mathcal{W}^{*,\mathrm{f}}_a(\mathbb{R}^d) \mapsto \mathrm{M}(\Omega^*) \geq 0$ of ν. Here $\mathcal{W}^{*,\mathrm{f}}_a(\mathbb{R}^d)$ stands for the subset of $\mathcal{W}^*_a(\mathbb{R}^d)$ formed by finite loop configurations over $\mathbb{R}^d$; alternatively $\mathcal{W}^{*,\mathrm{f}}_a(\mathbb{R}^d)$ is the union $\bigcup\limits_{\Lambda} \mathcal{W}^*_a(\Lambda)$ of loop configuration spaces $\mathcal{W}^*_a(\Lambda)$ over all cubes $\Lambda = [-\mathrm{L}, \mathrm{L}]^d$. The Ruelle superstability bound with a constant ρ (see (1.1.14)) has the form

$$\mathrm{M}(\Omega^*) \leq \frac{\rho^{K(\Omega^*)}}{L(\Omega^*)} \tag{2.3.19}$$

and will follow from the representation

$$\mathrm{M}(\Omega^*) = \frac{\mathrm{z}^{K(\Omega^*)}}{L(\Omega^*)} \int_{\mathcal{W}^*_a(\mathbb{R}^d)} \exp\,\big[-h(\Omega^*|\mathbf{\Omega}^*)\big]\mu(\mathrm{d}\mathbf{\Omega}^*) \tag{2.3.20}$$

and assumption (1.1.14); see below.

2.4. Results on Infinite-Volume FK-DLR PMs and Gibbs States

Our results about classes $\mathfrak{K}$, $\mathfrak{F}$ and $\mathfrak{F}_+$ are summarized in the following theorems.

Theorem 6. *The class $\mathfrak{K}$ of FK-DLR PMs is non-empty. Moreover, the family of FK-DLR PMs μ_Λ is compact in the weak topology, and every limiting point μ for this family lies in $\mathfrak{K}$. Furthermore, the family of the Gibbs states φ_Λ is compact in the w^*-topology, and every limiting point for this family gives an element from $\mathfrak{F}_+$. The same is true for any family of the PMs $\mu_{\Lambda|\mathbf{x}(\Lambda^c)}$ and states $\varphi_{\Lambda|\mathbf{x}(\Lambda^c)}$ with configurations $\mathbf{x}(\Lambda^c) \in \mathcal{C}_a(\Lambda^c)$. Consequently, the set $\mathfrak{F}_+(\mathrm{z}, \beta)$ is non-empty.*

Theorem 7. *Set $d = 2$. Let μ be a PM from $\mathfrak{K}(\mathrm{z}, \beta)$. Then the corresponding FK-DLR functional $\varphi_\mu \in \mathfrak{F}(\mathrm{z}, \beta)$ is shift-invariant: $\forall$ square $\Lambda_0 \subset \mathbb{R}^2$, vector $s \in \mathbb{R}^2$ and operator $A \in \mathfrak{B}(\Lambda_0)$,*

$$\varphi_\mu(\mathrm{U}^{\Lambda_0}(s)A\mathrm{U}^{\Lambda_0}(-s)) = \varphi_\mu(A). \tag{2.4.1}$$

In terms of the corresponding infinite-volume RDMs R^{Λ_0}*:*

$$R^{\Lambda_0} = \mathrm{U}^{\Lambda_0}(-s) R^{\Lambda_0} \mathrm{U}^{\Lambda_0}(s). \tag{2.4.2}$$

Remark 2. *The statement of Theorem 7 is straightforward for the limit points* R^{Λ_0} *of the family* $\{R^{\Lambda_0}_{\Lambda}, \Lambda \nearrow \mathbb{R}^2\}$*, but requires a proof for the family* $\{R^{\Lambda_0}_{\Lambda,\mathbf{x}^{\mathrm{c}}(N)}\}$*.*

3. Proof of Theorems 1 and 6: A Compactness Argument

Let us fix a cube Λ_0 of sidelength $2L_0$ centered at $b = (\mathrm{b}^1, \dots, \mathrm{b}^d)$: $\Lambda_0 = \underset{1\leq j\leq d}{\times} [\mathrm{b}^j - L_0, \mathrm{b}^j + L_0]$. The first step in the proof is to verify that, as $\Lambda_0 \subset \Lambda$ and $\Lambda \nearrow \mathbb{R}^d$, the RDMKs $F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})}(\mathbf{x}_0, \mathbf{y}_0)$ (see (2.2.2)–(2.2.4), (2.2.12)–(2.2.14) and (2.2.16)) form a compact family in $C^0(\mathcal{C}_a(\Lambda_0) \times \mathcal{C}_a(\Lambda_0))$. (Recall, we have to work with pairs $(\mathbf{x}_0, \mathbf{y}_0)$ where cardinalities of $\mathbf{x}_0$ and $\mathbf{y}_0$ coincide: $\sharp\, \mathbf{x}_0 = \sharp\, \mathbf{y}_0$.) Note that Cartesian product $\mathcal{C}_a(\Lambda_0)$, the range of variables $\mathbf{x}_0$ and $\mathbf{y}_0$, is compact, as $\sharp\, \mathbf{x}_0$ and $\sharp\, \mathbf{y}_0$ are bounded, viz. $\sharp\, \mathbf{x}_0, \sharp\, \mathbf{y}_0 \leq v_0$ where $v_0 = v_0(\Lambda_0)$ is given by the upper integer part:

$$v_0 = \left\lceil (2L_0)^d (2\sqrt{d})^d / a^d \right\rceil.$$

As in [23,28,29], we employ the Ascoli–Arzela theorem, i.e., verify that the functions $F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})}(\mathbf{x}_0, \mathbf{y}_0)$ are uniformly bounded and equicontinuous.

Checking uniform boundedness is easy: from (2.2.2) and (2.2.17) one can see that, for $\sharp\, \mathbf{x}_0 = \sharp\, \mathbf{y}_0 = n$,

$$F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})}(\mathbf{x}_0, \mathbf{y}_0) \leq \overline{\rho}^n \ \text{ where } \ \overline{\rho} := \sum_{k\geq 1} \frac{\rho^k}{(2\pi k \beta)^{d/2}} \tag{3.1}$$

and $\rho = \mathrm{z}e^{\beta \mathrm{W}^-}$. It yields uniform boundedness in view of (1.1.14).

The argument for equi-continuity of RDMKs is based on uniform bounds upon the gradients $\nabla_x F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})}(\mathbf{x}_0, \mathbf{y}_0)$ and $\nabla_y F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})}(\mathbf{x}_0, \mathbf{y}_0)$, for $x \in \mathbf{x}_0$, $y \in \mathbf{y}_0$. Both cases are treated in a similar fashion; for definiteness, we consider gradients $\nabla_y F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})}(\mathbf{x}_0, \mathbf{y}_0)$, $\mathbf{x}_0, \mathbf{y}_0 \in \mathcal{C}_a(\Lambda_0)$.

It can be seen from representations (2.2.2)–(2.2.4) and (2.2.16) that there are two contributions into the gradient. The first contribution comes from varying the measure $\underline{\mathbb{P}}^*_{\mathbf{x}_0,\mathbf{y}_0}$. The second one emerges from varying the functional $\widehat{q}^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^{\mathrm{c}})}(\underline{\Omega}^*_0)$, more precisely, the numerator $\widehat{\Xi}^{\Lambda_0, \underline{\Omega}^*_0}_{\Lambda}[\Lambda \setminus \Lambda_0 | \mathbf{x}(\Lambda^{\mathrm{c}})]$ in (2.2.3). In fact, it is clear that the second contribution will come out only when we vary the term $\exp\left[-h\left(\underline{\Omega}^*_0 \vee \Omega^*_{\Lambda\setminus\Lambda_0} | \mathbf{x}(\Lambda^{\mathrm{c}})\right)\right]$ in (2.2.4). Of course, we are interested in varying a chosen point $y \in \mathbf{y}_0$.

Suppose that the particle configurations are $\mathbf{x}_0 = (x(1), \dots, x(n))$ and $\mathbf{y}_0 = (y(1), \dots, y(n))$ and the path configuration $\underline{\Omega}^*_0 = (\overline{\omega}^*(1), \dots, \overline{\omega}^*(n))$. For definiteness, assume that the involved permutation γ_n is identity and hence $\overline{\omega}^*(j) \in \overline{\mathcal{W}}^*_a(x(j), y(j))$, $1 \leq j \leq n$. To stress this fact, we will use the notation $\overline{\Omega}^*_0$ instead of $\underline{\Omega}^*_0$ and $\overline{\mathcal{W}}^*_a(\mathbf{x}_0, \mathbf{y}_0) = \underset{1\leq j\leq n}{\times} \overline{\mathcal{W}}^*_a(x(j), y(j))$ and $\overline{\mathbb{P}}^*_{\mathbf{x}_0,\mathbf{y}_0}(\mathrm{d}\overline{\Omega}^*_0) = \underset{1\leq j\leq n}{\times} \overline{\mathbb{P}}^*_{x(j),y(j)}(\mathrm{d}\overline{\omega}^*(j))$ in place of $\overline{\mathcal{W}}^*_a(\mathbf{x}_0, \mathbf{y}_0)$ and $\underline{\mathbb{P}}^*_{\mathbf{x}_0,\mathbf{y}_0}(\mathrm{d}\underline{\Omega}^*_0)$, respectively. For $\mathbf{x}_0 = \mathbf{y}_0$, we obtain $\overline{\mathcal{W}}^*_a(\mathbf{x}_0, \mathbf{x}_0) = \underset{1\leq j\leq n}{\times} \mathcal{W}^*_a(x(j))$ and $\overline{\mathbb{P}}^*_{\mathbf{x}_0,\mathbf{y}_0}(\mathrm{d}\overline{\Omega}^*_0) = \underset{1\leq j\leq n}{\times} \mathbb{P}^*_{x(j)}(\mathrm{d}\omega^*(j))$, and $\overline{\Omega}^*_0$ becomes a configuration of loops $\omega^*(j) \in \mathcal{W}_a^*(x(j))$. Effectively, we have to analyze the gradient $\nabla_{y(j)}$ of the following expression:

$$\begin{aligned}&\int_{\overline{\mathcal{W}}^*_a(\mathbf{x}_0,\mathbf{y}_0)} \exp\,\Big[-h\Big(\overline{\Omega}^*_0|\Omega^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{x}(\Lambda^{\mathrm{c}})\Big)\Big]\,\overline{\mathbb{P}}^*_{\mathbf{x}_0,\mathbf{y}_0}(\mathrm{d}\overline{\Omega}^*_0)\\ &\quad=\int_{\overline{\mathcal{W}}^*_a(\mathbf{x}_0,\mathbf{x}_0)} \exp\,\Big[-h\Big(\overline{\Omega}^*_0+\overline{\mathrm{Z}}^*|\Omega^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{x}(\Lambda^{\mathrm{c}})\Big)\Big]\\ &\qquad\times\prod_{1\le i\le n}\exp\,\Big\{-|x(i)-y(i)|^2/[2k(\omega^*(i))\beta]\Big\}\overline{\mathbb{P}}^*_{\mathbf{x}_0,\mathbf{x}_0}(\mathrm{d}\overline{\Omega}^*_0).\end{aligned}\tag{3.2}$$

Here, $\overline{\mathrm{Z}}^*$ is a collection of linear paths: $\overline{\mathrm{Z}}^* = (\overline{\zeta}^*(1),\dots,\overline{\zeta}^*(n))$ where

$$\overline{\zeta}^*(i):\mathtt{t}\in\big[0,k(\omega^*(i))\beta\big]\mapsto\frac{\mathtt{t}}{k(\omega^*(i))\beta}(y(i)-x(i)),\ \ 1\le i\le n.$$

The first aforementioned gradient contribution emerges when we differentiate the term $\exp\Big\{-|x(j)-y(j)|^2/[2k(\omega^*(j))\beta]\Big\}$, the second while doing $\exp\Big[-h\Big(\omega^*(j)+\overline{\zeta}^*(j)|(\overline{\Omega}^*_0+\overline{\mathrm{Z}}^*)_{\setminus j}\vee\overline{\Omega}^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{x}(\Lambda^{\mathrm{c}})\Big)\Big]$. Here, $(\overline{\Omega}^*_0+\overline{\mathrm{Z}}^*)_{\setminus j}$ stands for a reduced path configuration $\overline{\Omega}^*_0+\overline{\mathrm{Z}}^*$, with path $\omega^*(j)+\overline{\zeta}(j)$ removed. Consequently, we obtain that the gradient of (3.2) has the form

$$\begin{aligned}&\int\limits_{\overline{\mathcal{W}}^*_a(\mathbf{x}_0,\mathbf{x}_0)}\Big[-\nabla_{y(j)}h(\omega^*(j)+\overline{\zeta}^*(j)|(\overline{\Omega}^*_0+\overline{\mathrm{Z}}^*)_{\setminus j}\vee\overline{\Omega}^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{x}(\Lambda^{\mathrm{c}}))\\ &\qquad-\frac{\nabla_{y(j)}|x(j)-y(j)|^2}{2k(\omega^*(y(j)))}\Big]\exp\,\Big[-h\Big(\overline{\Omega}^*_0+\overline{\mathrm{Z}}^*|\Omega^*_{\Lambda\setminus\Lambda_0}\vee\mathbf{x}(\Lambda^{\mathrm{c}})\Big)\Big]\\ &\qquad\times\prod_{1\le i\le n}\exp\,\Big\{-|x(i)-y(i)|^2/[2k(\omega^*(i))\beta]\Big\}\overline{\mathbb{P}}^*_{\mathbf{x}_0,\mathbf{x}_0}(\mathrm{d}\overline{\Omega}^*_0).\end{aligned}\tag{3.3}$$

Given $\widetilde{j}=1,\dots,n$, write $k(\widetilde{j})$ for $k(\omega^*(\widetilde{j}))$ and $\omega^*_{\widetilde{j}}$ and $\overline{\zeta}^*_{\widetilde{j}}$ for $\omega^*(\widetilde{j})$ and $\overline{\zeta}^*(\widetilde{j})$. Then the first gradient in (3.3) equals

$$\begin{aligned}&-\sum_{0\le l<k(j)}\nabla_{y(j)}\int_0^\beta\Big\{\sum_{l<l'<k(j)}V\Big(\big|(\omega^*_j+\overline{\zeta}^*_j)(\mathtt{t}+l\beta)-(\omega^*_j+\overline{\zeta}^*_j)(\mathtt{t}+l'\beta)\big|\Big)\\ &\quad+\sum_{\substack{1\le j'\le n:j'\neq j\\0\le l'<k(j')}}V\Big(\big|(\omega^*_j+\overline{\zeta}^*_j)(\mathtt{t}+l\beta)-(\omega^*_{j'}+\overline{\zeta}^*_{j'})(\mathtt{t}+l'\beta)\big|\Big)\\ &\quad+\sum_{\substack{\widetilde{\omega}^*\in\Omega^*_{\Lambda\setminus\Lambda_0}\\0\le\widetilde{l}<k(\widetilde{\omega}^*)}}V\Big(\big|(\omega^*_j+\overline{\zeta}^*_j)(\mathtt{t}+l\beta)-\widetilde{\omega}^*(\mathtt{t}+\widetilde{l}\beta)\big|\Big)\\ &\quad+\sum_{u\in\mathbf{x}(\Lambda^{\mathrm{c}})}V\Big(\big|(\omega^*_j+\overline{\zeta}^*_j)(\mathtt{t})-y\big|\Big)\Big\}\mathrm{d}\mathtt{t}:=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}.\end{aligned}$$

For the last three contributions we have:

$$\begin{aligned}
&\Big|\mathrm{II}+\mathrm{III}+\mathrm{IV}\Big| \le \big|\mathrm{II}\big|+\big|\mathrm{III}\big|+\big|\mathrm{IV}\big| \le \sum_{0\le l<k(j)} \int_0^\beta \frac{2(\mathtt{t}+l\beta)}{k(j)\beta}\\
&\times\Bigg\{ \sum_{\substack{1\le j'\le n: j'\neq j\\ 0\le l'<k(j')}} \Big|V'\big(\big|(\omega_j^*+\overline{\zeta}_j^*)(\mathtt{t}+l\beta)-(\omega_{j'}^*+\overline{\zeta}_{j'}^*)(\mathtt{t}+l'\beta)\big|\big)\Big|\\
&\quad+\sum_{\substack{\widetilde{\omega}^*\in\Omega^*_{\Lambda\setminus\Lambda_0}\\ 0\le\widetilde{l}<k(\widetilde{\omega}^*)}} \Big|V'\big(\big|(\omega_j^*+\overline{\zeta}_j^*)(\mathtt{t}+l\beta)-\widetilde{\omega}^*(\mathtt{t}+\widetilde{l}\beta)\big|\big)\Big|\\
&\quad+\sum_{u\in\mathbf{x}(\Lambda^c)} \Big|V'\big(\big|(\omega_j^*+\overline{\zeta}_j^*)(\mathtt{t}+l\beta)-y\big|\big)\Big|\Bigg\}\mathrm{d}\mathtt{t} \le \mathrm{W}^{(1)}k(j)\beta.
\end{aligned}$$

The quantity $\mathrm{W}^{(1)}$ is given in (1.1.3b). For the first contribution:

$$\begin{aligned}
\big|\mathrm{I}\big| \le &\sum_{0\le l<l'<k(j)} \int_0^\beta \frac{2(\mathtt{t}+l\beta)+2(\mathtt{t}+l'\beta)}{k(j)\beta}\\
&\times\Big|V'\big(\big|(\omega_j^*+\overline{\zeta}_j^*)(\mathtt{t}+l\beta)-(\omega_j^*+\overline{\zeta}_j^*)(\mathtt{t}+l'\beta)\big|\big)\Big| \le 2\mathrm{W}^{(1)}k(j)\beta.
\end{aligned}$$

Thus,

$$\Big|\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}\Big| \le 3W^{(1)}k(j)\beta.$$

The integral of the gradient $\nabla_{y(j)}|x(j)-y(j)|^2$ in (3.3) does not exceed a constant C. Hence,

$$\text{the norm of (3.3)} \le \lceil v(\Lambda_0)\rceil!\Big\{C\vee\big[3\mathrm{W}^{(1)}\beta\overline{\rho}\big]\Big\}^{\lceil v(\Lambda_0)\rceil}. \tag{3.4}$$

This shows equicontinuity of functions $F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}(\mathbf{x}_0,\mathbf{y}_0)$.

Hence, the family of RDMKs $\{F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}\}$ is compact in space $C^0(\mathcal{C}_a(\Lambda_0)\times\mathcal{C}_a(\Lambda_0))$. Let F^{Λ_0} be a limit-point as $\Lambda\nearrow\mathbb{R}^d$. Then we have the Hilbert–Schmidt convergence

$$\lim_{k\to\infty}\int\limits_{\mathcal{C}(\Lambda_0)\times\mathcal{C}(\Lambda_0)}\Big[F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}(\mathbf{x}_0,\mathbf{y}_0)-F^{\Lambda_0}(\mathbf{x}_0,\mathbf{y}_0)\Big]^2\mathrm{d}^{\Lambda_0}\mathbf{x}_0\,\mathrm{d}^{\Lambda_0}\mathbf{y}_0=0.$$

Consequently, the RDM $R^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}$ in $\mathcal{H}(\Lambda_0)$ converges to the infinite-volume RDM R^{Λ_0} determined by the kernel F^{Λ_0}, in the Hilbert-Schmidt norm:

$$\Big\|R^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}-R^{\Lambda_0}\Big\|_{\mathrm{HS}}\to 0. \tag{3.5}$$

As was mentioned, applying Lemma 1 from [24] (see also Lemma 1.5 from [23]), we obtain the trace-norm convergence:

$$\Big\|R^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}-R^{\Lambda_0}\Big\|_{\mathrm{tr}}\to 0. \tag{3.6}$$

Invoking a standard diagonal process implies that the sequence of states $\varphi_{\Lambda|\mathbf{x}(\Lambda^c)}$ is w*-compact.

Alongside with the above argument, one can establish that the PMs $\mu_{\Lambda|\mathbf{x}(\Lambda^c)}$ form a compact family as $\Lambda\nearrow\mathbb{R}^2$. More precisely, we would like to show that for all given cube Λ_0, the family of PMs $\mu^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}$ on $(\mathcal{W}^*_a(\Lambda_0),\mathfrak{W}(\Lambda_0))$ is compact. To this end, it suffices to check that the family $\{\mu^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}\}$ is tight as the Prokhorov theorem will then guarantee compactness.

Following an argument from [23], tightness is a consequence of two facts.

(a) The reference measure $\mathrm{d}^{\Lambda_0}\Omega_0^*$ on W^{Λ_0} (see Definition 1 (vii)) is supported by loop configurations with the standard continuity modulus $\sqrt{2\epsilon\ln(1/\epsilon)}$.

(b) The PDF $f_{\Lambda|\mathbf{x}(\Lambda^c)}(\Omega^*) = \dfrac{\mu_{\Lambda|\mathbf{x}(\Lambda^c)}(\mathrm{d}\Omega^*)}{\mathrm{d}^\Lambda\Omega^*}$ (cf. (2.2.6)) is bounded from above by a constant similar to the RHS of (3.1).

As a result, the family of limit-point PMs $\{\mu^{\Lambda_0} : \Lambda_0 \subset \mathbb{R}^2\}$ has the compatibility property and therefore satisfies the assumptions of the Kolmogorov theorem. This implies that there exists a unique PM μ on $(\mathcal{W}_a^*(\mathbb{R}^d), \mathfrak{W}(\mathbb{R}^d))$ such that the restriction of μ on the sigma-algebra $\mathfrak{W}(\Lambda_0)$ coincides with μ^{Λ_0}.

The fact that μ is an FK-DLR PM follows from the above construction. Hence, each limit-point state φ falls in class $\mathfrak{F}_+(\mathrm{z}, \beta)$. This completes the proof of Theorems 1 and 6.

4. Proof of Theorem 2: A Tuned-Shift Argument

From now on, we suppose that $d = 2$ and assume the conditions of Theorem 2. In view of Formulas (2.3.5)–(2.3.8) relating an FK-DLR functional φ to an FK-DLR measure μ, it suffices to verify

Theorem 8. *Any FK-DLR PM μ is translation invariant: for all $s = (\mathrm{s}^1, \mathrm{s}^2) \in \mathbb{R}^2$, square $\Lambda_0 = [-L_0, L_0]^{\times 2}$ and event $\mathcal{D} \in \mathcal{W}^*(\mathbb{R}^2)$ localised in Λ_0 (i.e., belonging to a sigma-algebra $\mathfrak{W}(\Lambda_0)$),*

$$\mu(\mathrm{S}(s)\mathcal{D}) = \mu(\mathcal{D}).$$

The proof of Theorem 8 is based on a modification of an argument developed in [11–13]. We want to stress that the paper [13] treating some classes of (Gibbsian) RMPPs does not cover our situation because a number of the assumptions used in [13] are (unfortunately) not fulfilled here. Specifically, the condition (2.2) from [13] does not hold in our situation, as well as conditions specifying what is called a bpsi-function on P. 704 of [13]. (In short, the paper [13] employs an approach based on sup-norm conditions whereas the situation under consideration in this paper requires the use of integral-type norms.) The aforementioned modification requires that we use (and inspect) the construction from [12] for particle configurations arising as t-sections of loop and path configurations at a given time point $\mathrm{t} \in [0, \beta]$.

Because the argument in the proof does not depend on the direction of the vector s, we will assume that $s = (\mathrm{s}, 0)$ lies along the horizontal axis. Also, due to the group property, we can assume that $\mathrm{s} \in (0, 1/2)$. By using constructions developed in [12,13,19], the assertion of Theorem 8 can be deduced from

Theorem 9. *Let μ be an FK-DLR PM, Λ_0 be a square $[-L_0, L_0]^{\times 2}$ and an event $\mathcal{D} \subset \mathcal{W}_a^*(\mathbb{R}^2)$ be given, localized in Λ_0: $\mathcal{D} \in \mathfrak{W}(\Lambda_0)$. Then*

$$\mu(\mathrm{S}^*(s)\mathcal{D}) + \mu(\mathrm{S}^*(-s)\mathcal{D}) - 2\mu(\mathcal{D}) \geq 0. \tag{4.1}$$

For the proof of Theorem 9, we employ a strategy essentially mimicking the one from [11–13], particularly [12]. Consequently, we will follow the scheme from [12] rather closely, although, as was said earlier, we introduce considerable alterations. For a given (large) L, we work with the squares $\Lambda = \Lambda(\mathrm{L})$ and Λ_0 where

$$\Lambda := [-\mathrm{L}, \mathrm{L}]^2 \supset [\mathrm{b}^1 - \mathrm{L}_0, \mathrm{b}^1 + \mathrm{L}_0] \times [\mathrm{b}^2 - \mathrm{L}_0, \mathrm{b}^2 + \mathrm{L}_0] =: \Lambda_0. \tag{4.2}$$

We write the terms $\mu(\mathrm{S}(\pm s)\mathcal{D})$ and $\mu(\mathcal{D})$ as integrals of conditional expectations relative to the sigma-algebra $\mathfrak{W}(\Lambda^c)$:

$$\begin{aligned} &\int_{\mathcal{W}_a^*(\mathbb{R}^2)} \mu(\mathrm{d}\boldsymbol{\Omega}^*_{\Lambda^c}) \mathbf{1}\Big(\boldsymbol{\Omega}^*_{\Lambda^c} \in W_a\,(\Lambda^c)\Big) \\ &\quad \times \int_{\mathcal{W}_a^*(\Lambda)} \mathrm{d}\Omega^*_\Lambda \mathbf{1}\Big(\Omega^*_\Lambda \in \mathrm{S}(\pm s) D\Big) \frac{\mathrm{z}^{K(\Omega^*_\Lambda)}}{L(\Omega^*_\Lambda)} \exp\big[-h(\Omega^*_\Lambda | \boldsymbol{\Omega}^*_{\Lambda^c}) \big] \end{aligned} \tag{4.3}$$

(the case of $\mu(\mathcal{D})$ is recovered at $s = 0$, with $\mathtt{S}(0) = \mathrm{Id}$.)

Furthermore, again as in [11,13], we employ maps $\mathtt{T}^{\pm}_{\mathrm{L}} = \mathtt{T}^{\pm}_{\mathrm{L,L_0}}(s) : \mathcal{W}^*(\mathbb{R}^2) \to \mathcal{W}^*_a(\mathbb{R}^2)$. (The symbol used in [11,13] is $\mathfrak{T}$ instead of $\mathtt{T}$. The idea of using maps $\mathtt{T}^{\pm}_{\mathrm{L}}$ goes back to [9,10].) These are applied to the concatenated loop configuration $\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c}$ in the expressions from Equation (4.3), in the corresponding case of shift $\mathtt{S}(\pm s)$. Important properties of maps $\mathtt{T}^{\pm}_{\mathrm{L}}$ are:

(i) The maps $(\Omega^*_\Lambda, \mathbf{\Omega}^*_{\Lambda^c}) \mapsto \mathtt{T}^{\pm}_{\mathrm{L}}(\Omega^*_\Lambda, \mathbf{\Omega}^*_{\Lambda^c})$ are one-to-one, and a number of 'nice' properties hold true when the loop configuration $\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c}$ lies in a 'good' set $\mathcal{G}_{\mathrm{L}} \subset \mathcal{W}^*_a(\mathbb{R}^2)$. (Viz., for $\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c} \in \mathcal{G}_{\mathrm{L}}$ the loops from $\Omega^*_\Lambda \cap \mathcal{W}^*_a(\Lambda_0)$ will not interact with loops from $\mathbf{\Omega}^*_{\Lambda^c}$.) The set $\mathcal{G}_{\mathrm{L}}$ carries asymptotically a full measure as $\mathrm{L} \to \infty$. See below.

(ii) For a 'good' loop configuration $\mathbf{\Omega}^* = \mathbf{\Omega}^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c} \in \mathcal{G}_{\mathrm{L}}$ over $\mathbb{R}^2$, the 'external' part $\mathbf{\Omega}^*_{\Lambda^c}$ is preserved under $\mathtt{T}^{\pm}_n$. In other words, the maps are non-trivial only on the part $\mathbf{\Omega}^*_\Lambda$ (although the way $\mathbf{\Omega}^*_\Lambda$ is transformed depends upon $\mathbf{\Omega}^*_{\Lambda^c}$ (and on $\mathbf{\Omega}^*_\Lambda$, of course)). For that reason, we will often address $\mathtt{T}^{\pm}_{\mathrm{L}}$ as a 'tuned' shift $\mathbf{\Omega}^* \mapsto \widetilde{\mathbf{\Omega}}^* = (\mathtt{T}^{\pm}_{\mathrm{L}}\mathbf{\Omega}^*_\Lambda) \vee \mathbf{\Omega}^*_{\Lambda^c}$ or, dealing with a pair $(\Omega^*_\Lambda, \mathbf{\Omega}^*_{\Lambda^c}) \in \mathcal{W}^*(\Lambda, \Lambda^c)$,

$$\Omega^*_\Lambda \mapsto \widetilde{\Omega}^*_\Lambda = \mathtt{T}^{\pm}_{\mathrm{L}}\Omega^*_\Lambda \in \mathcal{W}^*_a(\Lambda). \tag{4.4}$$

With this agreement:

(iii) The transformation (4.4) preserves the cardinality: $\sharp\,\Omega^*_\Lambda = \sharp\,\widetilde{\Omega}^*_\Lambda$ and transforms a loop $\omega^* \in \Omega^*_\Lambda$ as $\omega^* \mapsto \widetilde{\omega}^*$ where $k(\widetilde{\omega}^*) = k(\omega^*)$. Consequently, functionals K and L are preserved: $K(\widetilde{\Omega}^*) = K(\Omega^*)$ and $L(\widetilde{\Omega}^*) = L(\Omega^*)$. Next, $\forall\, \mathtt{t} \in [0, k(\omega^*)\beta]$, point $\widetilde{\omega}^*(\mathtt{t}) \in \mathbb{R}^2$ is obtained as a 'tuned shift'

$$\widetilde{\omega}^*(\mathtt{t}) = \omega^*(\mathtt{t}) \pm s\mathtt{R}^{\pm}_{\mathrm{L}}\Big[\omega^*; \mathtt{t}; \{\Omega^*_\Lambda\}(\mathtt{t}) \cup \{\mathbf{\Omega}^*_{\Lambda^c}\}(\mathtt{t})\Big]; \tag{4.5}$$

see below. We stress that the argument of function $\mathtt{R}^{\pm}_{\mathrm{L}}$ consists of a loop $\omega^* \in \mathcal{W}^*_a$, a time point $\mathtt{t} \in [0, k(\omega^*)\beta]$ and the $\mathtt{t}$-section $\{\Omega^*_\Lambda\}(\mathtt{t}) \cup \{\mathbf{\Omega}^*_{\Lambda^c}\}(\mathtt{t})) = \{\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c}\}(\mathtt{t}) \in \mathcal{C}_a(\mathbb{R}^2)$ of a loop configuration $\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c}$.

(iv) For brevity, let us omit henceforth the symbols $\pm$ whenever possible. The value $\mathtt{R}_{\mathrm{L}}\Big[\omega^*; \mathtt{t}; \{\Omega^*_\Lambda\}(\mathtt{t}) \cup \{\mathbf{\Omega}^*_{\Lambda^c}\}(\mathtt{t})\Big] \in [0,1]$. Moreover, when $\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c} \in \mathcal{G}_{\mathrm{L}}$ then for $\omega^* \in \Omega^*_\Lambda \cap \mathcal{W}^*_a(\Lambda_0)$ and $0 \le \mathtt{t} \le k(\omega^*)\beta$,

$$\mathtt{R}_{\mathrm{L}}\Big[\omega^*; \mathtt{t}; \{\Omega^*_\Lambda\}(\mathtt{t}) \cup \{\mathbf{\Omega}^*_{\Lambda^c}\}(\mathtt{t})\Big] \equiv 1, \;\; 0 \le \mathtt{t} \le k(\omega^*)\beta.$$

Consequently, in accordance with (4.5), for $\omega^* \in \mathcal{W}^*_a(x)$ with $x \in \Lambda_0$ and $\mathtt{t} \in [0, k(\omega^*)\beta]$ the point $\widetilde{\omega}^*(\mathtt{t}) = \omega^*(\mathtt{t}) + s$. Therefore, the loops ω^* from $\Omega^*_0 = \Omega^*_\Lambda \cap \mathcal{W}^*_a(\Lambda_0)$ are shifted intact by the amount s under the map (4.4). Consequently, the integral energy $h(\Omega^*_0)$ is not changed under tuned shifts.

(v) The set $\mathtt{S}(s)(\mathcal{D} \cap \mathcal{G}_{\mathrm{L}})$ will have a μ-measure close to that of $\mathtt{S}(s)\mathcal{D}$; moreover, the probability $\mu(\mathtt{S}(s)(\mathcal{D} \cap \mathcal{G}_{\mathrm{L}}))$ will be written in the form

$$\begin{aligned}\mu(\mathtt{S}(\pm s)(\mathcal{D} \cap \mathcal{G}_{\mathrm{L}})) = &\int_{\mathcal{W}^*_a(\mathbb{R}^2)} \mu(\mathrm{d}\mathbf{\Omega}^*_{\Lambda^c})\mathbf{1}\Big(\mathbf{\Omega}^*_{\Lambda^c} \in W_a\,(\Lambda^c)\Big) \\ &\times \int_{\mathcal{W}^*_a(\Lambda)} \mathrm{d}\Omega^*_\Lambda\; \mathbf{1}\Big(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c} \in \mathcal{G}_{\mathrm{L}} \cap \mathcal{D}\Big)\, \frac{\mathrm{z}^{K(\Omega^*_\Lambda)}}{L(\Omega^*_\Lambda)} \\ &\times J^{\pm}_{\mathrm{L}}(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c}) \exp\,\big[-h(\mathtt{T}^{\pm}_{\mathrm{L}}(s)\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^c})\big]\end{aligned} \tag{4.6}$$

where function $J^{\pm}_{\mathrm{L}} = J^{\pm}_{L,s}$ gives the Jacobian of transformation $\mathtt{T}^{\pm}_{\mathrm{L}}(s)$. By virtue of the properties above (cf. (i) and (iv)), the impact of $\mathtt{T}_{\mathrm{L}}$ upon the energy $h(\mathtt{T}_{\mathrm{L}}\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^c})$ will be felt through the loop configuration $\Omega^*_{\Lambda\setminus\Lambda_0} = \Omega^*_\Lambda \cap \mathcal{W}^*_a(\Lambda \setminus \Lambda^0)$ only. (More precisely, through a loop configuration $\Omega^*_{\Lambda\setminus\Lambda(\mathrm{R(L)})}$ where $\Lambda(\mathrm{R(L)}) = [-\mathrm{R(L)}, \mathrm{R(L)}]^{\times 2}$ and $\mathrm{R(L)} \nearrow \infty$ with L. See Equation (5.2) below.) Essentially, the same remains true about the Jacobian $J_{\mathrm{L}}(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c})$.

(vi) In fact, a detailed analysis shows that second-order incremental expressions

$$\Big[J^+_{\rm L}(\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}})J^-_{\rm L}(\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}})\Big]^{1/2} \tag{4.7}$$

and

$$\exp\;\Big[h(\mathtt{T}^+_{\rm L}(s)\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^{\rm c}})+h(\mathtt{T}^-_{\rm L}(s)\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^{\rm c}})-2h(\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^{\rm c}})\Big] \tag{4.8}$$

are close to 1. It turns out that this fact suffices for the assertion of Theorem 9.

Formally, Theorem 9 is derived from

Theorem 10. *For all $\delta>0$ there exists $\mathrm{L}^*_0=\mathrm{L}^*_0(\delta)>0$ such that for $\mathrm{L}\geq\mathrm{L}^*_0$*

$$\begin{aligned}(I)\qquad \mu(\mathcal{G}_{\rm L})=\int_{\mathcal{W}^*_a(\mathbb{R}^2)}&\mu({\rm d}\Omega^*_{\Lambda^{\rm c}})\,\mathbf{1}\Big(\Omega^*_{\Lambda^{\rm c}}\in\mathcal{W}^*_a(\Lambda^{\rm c})\Big)\\ \times\int_{\mathcal{W}^*_a(\Lambda)}&{\rm d}\Omega^*_\Lambda\,\mathbf{1}\Big(\Omega^*_\Lambda\vee\Omega^*_{\Lambda^{\rm c}}\in\mathcal{G}_{\rm L}\Big)\\ &\times\frac{\mathrm{z}^{K(\Omega^*_\Lambda)}}{L(\Omega^*_\Lambda)}\;\exp\;\big[-h(\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^{\rm c}})\big]\geq 1-\delta.\end{aligned} \tag{4.9}$$

(II) The probabilities $\mu(\mathtt{S}(\pm s)(\mathcal{D}\cap\mathcal{G}_{\rm L}))$ are represented in the form (4.6) with the following properties: $\forall\,\Omega^*_\Lambda\in\mathcal{W}^*_a(\Lambda)$, $\mathbf{\Omega}^*_{\Lambda^{\rm c}}\in\mathcal{W}^*_a(\Lambda^{\rm c})$ with $\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}}\in\mathcal{G}_{\rm L}$,

(IIIa) $\Big[J^+_{\rm L}(\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}})J^-_{\rm L}(\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}})\Big]^{1/2}\geq 1-\delta$;

(IIIb) $h(\mathtt{T}^+_{\rm L}(s)\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^{\rm c}})+h(\mathtt{T}^-_{\rm L}(s)\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^{\rm c}})-2h(\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^{\rm c}})\leq\delta$.

The proof of Theorem 10 is carried on in Sections 5–7.

Remark 3. *It is the pair of inequalities (IIIa), (IIIb) (together with the definition of the 'good' set $\mathcal{G}_{\rm L}$) where one crucially uses the fact that the physical dimension of the system equals 2.*

We now show how to deduce the statement of Theorem 9 from that of Theorem 10. Owing to Theorem 10 (I), (II), we can write:

$$\begin{aligned}&\text{the LHS of (4.1)}+3\delta\geq\mu(\mathtt{S}(s)(\mathcal{D}\cap\mathcal{G}_{\rm L}))+\mu(\mathtt{S}(-s)(\mathcal{D}\cap\mathcal{G}_{\rm L}))-2\mu(\mathcal{D}\cap\mathcal{G}_{\rm L})\\ &\quad=\int_{\mathcal{W}^*_a(\mathbb{R}^2)}\mu({\rm d}\mathbf{\Omega}^*_{\Lambda^{\rm c}})\mathbf{1}\Big(\mathbf{\Omega}^*_{\Lambda^{\rm c}}\in W_a\,(\Lambda^{\rm c})\Big)\int_{\mathcal{W}^*_a(\Lambda)}{\rm d}\Omega^*_\Lambda\,\mathbf{1}\Big(\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}}\in\mathcal{G}_{\rm L}\cap\mathcal{D}\Big)\\ &\qquad\times\frac{\mathrm{z}^{K(\Omega^*_\Lambda)}}{L(\Omega^*_\Lambda)}\Big\{J^+_{\rm L}(\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}})\exp\;\big[-h(\mathtt{T}^+_{\rm L}\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^{\rm c}})\big]\\ &\qquad+J^-_{\rm L}(\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}})\exp\;\big[-h(\mathtt{T}^-_{\rm L}\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^{\rm c}})\big]-2\exp\;\big[-h(\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^{\rm c}})\big]\Big\}.\end{aligned} \tag{4.10}$$

Next, by the AM/GM inequality, the RHS of (4.10) is no less than

$$\begin{aligned}&2\int_{\mathcal{W}^*_a(\mathbb{R}^2)}\mu({\rm d}\mathbf{\Omega}^*_{\Lambda^{\rm c}})\mathbf{1}\Big(\mathbf{\Omega}^*_{\Lambda^{\rm c}}\in W_a\,(\Lambda^{\rm c})\Big)\int_{\mathcal{W}^*_a(\Lambda)}{\rm d}\Omega^*_\Lambda\,\mathbf{1}\Big(\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}}\in\mathcal{G}_{\rm L}\cap\mathcal{D}\Big)\\ &\qquad\times\frac{\mathrm{z}^{K(\Omega^*_\Lambda)}}{L(\Omega^*_\Lambda)}\;\Big(\Big[J^+_{\rm L}(\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}})J^-_{\rm L}(\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}})\Big]^{1/2}\\ &\times\exp\Big\{-\big[h(\mathtt{T}^+_{\rm L}\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^{\rm c}})+h(\mathtt{T}^-_{\rm L}\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^{\rm c}})\big]/2\Big\}-\exp\;\big[-h(\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^{\rm c}})\big]\Big).\end{aligned} \tag{4.11}$$

Now, by virtue of Theorem 10 (I)–(III), the RHS of (4.11) is greater than or equal to

$$\begin{aligned} & 2[(1-\delta)e^{-\delta/2}-1]\int_{\mathcal{W}_a^*(\mathbb{R}^2)}\mu(\mathrm{d}\mathbf{\Omega}^*_{\Lambda^c})\mathbf{1}\Big(\mathbf{\Omega}^*_{\Lambda^c}\in W_a\,(\Lambda^c)\Big) \\ & \times\int_{\mathcal{W}_a^*(\Lambda)}\mathrm{d}\Omega^*_\Lambda\,\mathbf{1}\Big(\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^c}\in\mathcal{G}_{\mathrm{L}}\cap\mathcal{D}\Big)\frac{\mathrm{z}^{K(\Omega^*_\Lambda)}}{L(\Omega^*_\Lambda)}\exp\big[-h(\Omega^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^c})\big]-3\delta \\ & \quad=2[(1-\delta)e^{-\delta/2}-1]\mu(\mathcal{G}_{\mathrm{L}})-3\delta\geq 2[(1-\delta)e^{-\delta/2}-1]-3\delta. \end{aligned} \tag{4.12}$$

Since δ can be made arbitrarily small, we obtain the inequality (4.1).

5. Definition of Transformations $\mathtt{T}^{\pm}_{\mathrm{L}}$

As was said earlier, the maps $\mathbf{\Omega}^*\mapsto\mathtt{T}^{\pm}_{\mathrm{L}}\mathbf{\Omega}^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^c}$ are determined by transforming the $\mathtt{t}$-sections $\{\mathtt{T}^{\pm}_{\mathrm{L}}\mathbf{\Omega}^*_\Lambda\}(\mathtt{t})$ of the loop configuration $\mathbf{\Omega}^*_\Lambda$, for each $\mathtt{t}\in[0,\beta]$. Denoting by $T^{\pm}_{\mathrm{L}}=T^{\pm}_{\mathrm{L}}(\pm s)$ the map acting on particle configurations from $\mathcal{C}_a(\Lambda)$, we can write:

$$\{\mathtt{T}^{\pm}_{\mathrm{L}}\mathbf{\Omega}^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^c}\}(\mathtt{t})\quad=\{\mathtt{T}^{\pm}_{\mathrm{L}}\mathbf{\Omega}^*_\Lambda\}(\mathtt{t})\vee\mathbf{\Omega}^*_{\Lambda^c}(\mathtt{t})=\big(T^{\pm}_{\mathrm{L}}[\{\mathbf{\Omega}^*_\Lambda\}(\mathtt{t})]\big)\vee\mathbf{\Omega}^*_{\Lambda^c}(\mathtt{t}). \tag{5.1}$$

Again, we would like to stress that the way the $\mathtt{t}$-section $\{\mathbf{\Omega}^*_\Lambda\}(\mathtt{t})$ is transformed depends on $\{\mathbf{\Omega}^*_{\Lambda^c}\}(\mathtt{t})$, although $\{\mathbf{\Omega}^*_{\Lambda^c}\}(\mathtt{t})$ itself is not moving when $\mathbf{\Omega}^*\in\mathcal{G}_{\mathrm{L}}$.

More precisely, set:

$$\mathrm{R}(\mathrm{L})=\big(\log\log\,\mathrm{L}\big)^{3/4},\ \ \Lambda(\mathrm{R}(\mathrm{L}))=[-\mathrm{R}(\mathrm{L}),\mathrm{R}(\mathrm{L})]^{\times 2}, \tag{5.2}$$

and introduce yet another intermediate square

$$\overline{\Lambda}=[-\overline{\mathrm{L}},\overline{\mathrm{L}}]^{\times 2}\ \text{ where }\ \overline{\mathrm{L}}=\mathrm{L}-\mathrm{L}^{3/4}. \tag{5.3}$$

The transformed particle configuration $T^{\pm}_{\mathrm{L}}[\{\mathbf{\Omega}^*_\Lambda\}(\mathtt{t})]\in\mathcal{C}_a(\Lambda)$ is formed by points $\widetilde{\omega}^{*\pm}_{\mathrm{L}}(l\beta+\mathtt{t})$ obtained, as a result of shifts in the (positive) horizontal direction, from the points $\omega^*(l\beta+\mathtt{t})$ where $\mathtt{t}\in[0,\beta]$, $l=0,\dots,k(\omega^*)-1$ and $\omega^*\in\mathbf{\Omega}^*_\Lambda$:

$$\widetilde{\omega}^{*\pm}_{\mathrm{L}}(l\beta+\mathtt{t})=\omega^*(l\beta+\mathtt{t})\pm\mathrm{p}_{\mathrm{L}}(\omega^*(l\beta+\mathtt{t}))\times s. \tag{5.4}$$

Here, the scalar value $\mathrm{p}_{\mathrm{L}}(\omega^*(l\beta+\mathtt{t}))\geq 0$ depends on particle configurations $\{\mathbf{\Omega}^*_\Lambda\}(\mathtt{t})$ and $\{\mathbf{\Omega}^*_{\Lambda^c}\}(\mathtt{t})$ and are constructed recursively; cf. [12]. When $\omega^*(l\beta+\mathtt{t})\in\Lambda\setminus\overline{\Lambda}$, we have that

$$\mathrm{p}_{\mathrm{L}}(\omega^*(l\beta+\mathtt{t}))=0\ \text{ and }\ \widetilde{\omega}^{*\pm}_{\mathrm{L}}(l\beta+\mathtt{t})=\omega^*(l\beta+\mathtt{t}).$$

In other words, a loop $\omega^*\in\mathbf{\Omega}^*$ is affected only at points $\omega^*(\mathtt{t})$ lying in $\overline{\Lambda}$.

In the course of construction of values $\mathrm{p}_{\mathrm{L}}(\omega^*(l\beta+\mathtt{t}))$, we employ the function $u\in[0,\infty)\mapsto\tau_{\mathrm{L}}(u)$ determined as follows:

$$\tau_{\mathrm{L}}(u)=\begin{cases}1, & 0\leq u\leq\mathrm{R}(\mathrm{L}),\\ 1-\dfrac{Q(u-\mathrm{R}(\mathrm{L}))}{Q(\mathrm{L}-\mathrm{R}(\mathrm{L}))}, & \mathrm{R}(\mathrm{L})\leq u\leq\overline{L},\\ 0, & u\geq\overline{L},\end{cases} \tag{5.5}$$

where, in turn, (Function τ_{L} was introduced in [9,10] and has been repeatedly used in the literature.)

$$Q(u)=\int_0^u q(v)\mathrm{d}v,\ \ \text{with}\ \ q(v)=\frac{1}{1\vee v|\log v|}. \tag{5.6}$$

The values $\mathtt{p}(\omega^*(l\beta+\mathtt{t})) = \mathtt{p}_{\mathrm{L}}(\omega^*(l\beta+\mathtt{t}))$ are related to results of a series of minimizations, over points $\omega^*(l\beta+\mathtt{t}) \in \{\mathbf{\Omega}^*\}(\mathtt{t}) \cap \overline{\Lambda}$, of subsequently introduced functions $\widetilde{t}^{\,(j)}(\,\cdot\,;\mathtt{t}) = \widetilde{t}^{\,(j)}_{\mathrm{L}}(\,\cdot\,;\mathtt{t})$. Here, j runs from 0 to $\#\left(\{\mathbf{\Omega}^*\}(\mathtt{t}) \cap \overline{\Lambda}\right)$ and the functions are

$$y \in \mathbb{R}^2 \mapsto \widetilde{t}^{\,(j)}(y;\mathtt{t}) \in [0,1], \qquad 0 \le j \le \sum_{\omega^* \in \mathbf{\Omega}^*} \sum_{0 \le l < k(\omega^*)} \mathbf{1}(\omega^*(l\beta+\mathtt{t}) \in \Lambda), \tag{5.7}$$

The value $j=0$ marks an initial function $t^{(0)}(\,\cdot\,;\mathtt{t})$ and the values $j \ge 1$ provide an ordering for points $\omega^*(l\beta+\mathtt{t})$ in the particle configuration $\{\mathbf{\Omega}^*\}(\mathtt{t}) \cap \overline{\Lambda}$. Let us stress that the functions $\widetilde{t}^{(j)}_{\mathrm{L}}(\,\cdot\,;\mathtt{t})$ involve (generally speaking) the whole $\mathtt{t}$-section $\{\mathbf{\Omega}^*\}(\mathtt{t})$.

The initial function in the series, $\widetilde{t}^{(0)}_{\mathrm{L}}(\,\cdot\,;\mathtt{t})$, does not depend on $\mathtt{t} \in [0,\beta]$ and is related to function $\tau = \tau_{\mathrm{L}}$ from (5.5):

$$\widetilde{t}^{(0)}(y;\mathtt{t}) := \tau\left(|y|_{\mathrm{m}}\right). \tag{5.8}$$

The definition of the next function, $\widetilde{t}^{(1)}_{\mathrm{L}}(\,\cdot\,;\mathtt{t})$, involves a (multiple) minimum of auxiliary functions $m_{x,0}$, over the points $x = \omega^*(l\beta+\mathtt{t})$ from the particle configuration $\{\mathbf{\Omega}^*\}(\mathtt{t}) \cap \overline{\Lambda}^{\mathrm{c}}$:

$$\widetilde{t}^{(1)}(y;\mathtt{t}) = \widetilde{t}^{(0)}(y;\mathtt{t}) \wedge \widetilde{m}^{(0)}(y;\mathtt{t}) \tag{5.9}$$

where

$$\widetilde{m}^{(0)}(y;\mathtt{t}) = \widetilde{m}^{(0)}_{\mathrm{L}}(y;\mathtt{t}) = \bigwedge_{l:\,\omega^*(l\beta+\mathtt{t}) \in \{\mathbf{\Omega}^*\}(\mathtt{t}) \cap \overline{\Lambda}^{\mathrm{c}}} m_{\omega^*(l\beta+\mathtt{t}),0}\,(y). \tag{5.10}$$

Here and below, following [9,10,12], the family of auxiliary functions $y \in \mathbb{R}^2 \mapsto m_{x,\mathtt{u}}(u)$ is used, with values in $[0,1) \cup \{+\infty\}$, where $x \in \mathbb{R}^2$, $\mathtt{u} \in [0,1)$. These functions are introduced as follows:

$$m_{x,\mathtt{r}}(y) := \begin{cases} \mathtt{u}, & \mathtt{h}_{x,\mathtt{u}}\mathtt{c}_f > 1/2, \\ \mathtt{u} + \mathtt{h}_{x,\mathtt{u}} f(x-y) + \infty \cdot \mathbf{1}(f(x-y)=1), & \mathtt{h}_{x,\mathtt{r}}\mathtt{c}_f \le 1/2. \end{cases} \tag{5.11}$$

In turn, $f = f_\epsilon$ is a chosen C^1-function $\mathbb{R}^2 \to [0,1]$, with

$$f(v) = 0 \;\text{ when }\; |v| < a \;\text{ and }\; f(v) = 1 \;\text{ when }\; |v| > a + 2\epsilon,$$

and

$$\mathtt{c}_f = \max\left[|\nabla f(v)|,\ v \in \mathbb{R}^2\right]. \tag{5.12}$$

The value ϵ is selected for given $\mathtt{z}$ and β satisfying (1.1.14) and should be small enough, guaranteeing smallness of quantities introduced below. Finally,

$$\mathtt{h}_{x,\mathtt{u}} := |\tau(|x|_{\mathrm{m}} - \epsilon - a/2) - \mathtt{u}\,| \tag{5.13}$$

is another auxiliary parameter.

Pictorially speaking, the function $y \in \mathbb{R}^2 \mapsto \widetilde{m}^{(0)}(y;\mathtt{t})$ indicates by how much a particle (i.e., a circle of diameter a) placed at the reference point y could be moved (under adopted arrangements) in presence of hard-core particles placed at points $\omega^*(l\beta+\mathtt{t}) \in \{\mathbf{\Omega}^*\}(\mathtt{t}) \cap \overline{\Lambda}^{\mathrm{c}}$. Consequently, $\widetilde{t}^{(1)}_{\mathrm{L}}(y;\mathtt{t})$ indicates how much a movement by quantity $\widetilde{t}^{(0)}_{\mathrm{L}}(y;\mathtt{t})$ should be reduced in presence of hard-core particles at $\omega^*(l\beta+\mathtt{t}) \in \{\mathbf{\Omega}^*\}(\mathtt{t}) \cap \overline{\Lambda}^{\mathrm{c}}$. We then look for the minimum of $\widetilde{t}^{(1)}_{\mathrm{L}}(\,\cdot\,;\mathtt{t})$ over the particle configuration $\{\mathbf{\Omega}^*\}(\mathtt{t}) \cap \overline{\Lambda}$ and set:

$$\begin{aligned} \mathtt{p}^1 = \mathtt{p}^1_{\mathrm{L}} &= \min\left[\widetilde{t}^{(1)}_{\mathrm{L}}(y;\mathtt{t}):\ y \in \{\mathbf{\Omega}^*\}(\mathtt{t}) \cap \overline{\Lambda}\right], \\ P^1 = P^1_{\mathrm{L}} &= \arg\min\left[\widetilde{t}^{(1)}_{\mathrm{L}}(y;\mathtt{t}):\ y \in \{\mathbf{\Omega}^*\}(\mathtt{t}) \cap \overline{\Lambda}\right]. \end{aligned} \tag{5.14}$$

If the minimum is attained at more than one point in $\{\mathbf{\Omega}^*\}(\mathtt{t})\cap\overline{\Lambda}$, we list all these points: $P^1,\ldots,P^{\kappa_1}$ (in any order). The value $\mathtt{p}^1$ is assigned to each of those points as $\mathtt{p}(P^j)$:

$$\begin{aligned}\mathtt{p}(\omega^*(l\beta+\mathtt{t}))=\mathtt{p}^1,\ \text{ if }\ \omega^*\in\mathbf{\Omega}^*,0\le l<k(\omega^*),\\ \omega^*(l\beta+\mathtt{t})\in\overline{\Lambda}\ \text{ and }\ \widetilde{t}^{(1)}(\omega^*(l\beta+\mathtt{t});\mathtt{t})=\mathtt{p}^1.\end{aligned}\tag{5.15}$$

The value $\mathtt{p}^1$ and the position P^1 (or the positions P^1, P^2, …, P^{κ_1}) are taken into account in the definition of the next function $y\in\mathbb{R}^2\mapsto\widetilde{t}^{(2)}(y;\mathtt{t})$:

$$\widetilde{t}^{(2)}(y;\mathtt{t})=\widetilde{t}^{(1)}(y;\mathtt{t})\wedge m_{P^1,\mathtt{p}^1\times\mathtt{s}}(y)\ldots\wedge m_{P^{\kappa_1},\mathtt{p}^1\times\mathtt{s}}(y)=\widetilde{t}^{(0)}(y;\mathtt{t})\wedge\widetilde{m}^{(1)}(y;\mathtt{t}).\tag{5.16}$$

Here, $\widetilde{m}^{(1)}(y;\mathtt{t})=\widetilde{m}^{(1)}_{\mathrm{L}}(y;\mathtt{t})$ is given by

$$\widetilde{m}^{(1)}(y;\mathtt{t})=\widetilde{m}^{(0)}(y;\mathtt{t})\wedge\left(\bigwedge_{l:\,\omega^*(l\beta+\mathtt{t})\in\{\mathbf{\Omega}^*\}^1(\mathtt{t})\cap\overline{\Lambda}}m_{\omega^*(l\beta+\mathtt{t}),\mathtt{p}^1\times\mathtt{s}}(y)\right)\tag{5.17}$$

and

$$\{\mathbf{\Omega}^*\}^1(\mathtt{t})=\left\{\omega^*(l\beta+\mathtt{t})\in\{\mathbf{\Omega}^*\}(\mathtt{t}):\ \widetilde{t}^{(1)}(y;\mathtt{t})=\mathtt{p}^1\right\}\tag{5.18}$$

yielding that

$$\{\mathbf{\Omega}^*\}^1(\mathtt{t})\cap\overline{\Lambda}=\{P^1,\ldots,P^{\kappa_s}\}.$$

(Recall, the initial shift-vector is $s=(\mathtt{s},0)$ where $\mathtt{s}\in[0,1/2)$.)

Pictorially speaking, the function $y\in\mathbb{R}^2\mapsto\widetilde{m}^{(1)}(y;\mathtt{t})$ indicates by how much a particle at point y could be moved when we take into account the particles placed at points $\omega^*(l\beta+\mathtt{t})\in\{\mathbf{\Omega}^*\}(\mathtt{t})\cap\overline{\Lambda}^{\mathrm{c}}$ (which do not move) and the particles placed at points $\omega^*(l\beta+\mathtt{t})\in\{\mathbf{\Omega}^*\}^1(\mathtt{t})\cap\overline{\Lambda}$ (which are moved by $\mathtt{p}^1$). Consequently, $\widetilde{t}^{(2)}(y;\mathtt{t})$ indicates how much a movement by quantity $\widetilde{t}^{(0)}(y;\mathtt{t})$ should be reduced in presence of hard-core particles at points $\omega^*(l\beta+\mathtt{t})\in\{\mathbf{\Omega}^*\}(\mathtt{t})\cap\overline{\Lambda}^{\mathrm{c}}$ and $\omega^*(l\beta+\mathtt{t})\in\{\mathbf{\Omega}^*\}^1(\mathtt{t})\cap\overline{\Lambda}$.

Next, we minimise the function $\widetilde{t}^{(2)}(\,\cdot\,;\mathtt{t})$ over the particle configuration $\big(\{\mathbf{\Omega}^*\}(\mathtt{t})\setminus\{\mathbf{\Omega}\}^1(\mathtt{t})\big)\cap\overline{\Lambda}$ and, like before, set:

$$\begin{aligned}\mathtt{p}^2=\min\left[\widetilde{t}^{(2)}_{\mathrm{L}}(y;\mathtt{t}):\ y\in\big(\{\mathbf{\Omega}^*\}(\mathtt{t})\setminus\{\mathbf{\Omega}\}^1(\mathtt{t})\big)\cap\overline{\Lambda}\right],\\ P^{\kappa_1+1}=\arg\min\left[\widetilde{t}^{(1)}(y;\mathtt{t}):\ y\in\big(\{\mathbf{\Omega}^*\}(\mathtt{t})\setminus\{\mathbf{\Omega}\}^1(\mathtt{t})\big)\cap\overline{\Lambda}\right].\end{aligned}\tag{5.19}$$

Again, if the minimum is shared by more than one point in the intersection $\big(\{\mathbf{\Omega}^*\}(\mathtt{t})\setminus\{\mathbf{\Omega}\}^1(\mathtt{t})\big)\cap\overline{\Lambda}$, we list all these points: P^{κ_1+1}, …, $P^{\kappa_1+\kappa_2}$ (in any order). As earlier, the value $\mathtt{p}^2$ is assigned to each of those points as $\mathtt{p}(P^j)$:

$$\begin{aligned}\mathtt{p}(\omega^*(l\beta+\mathtt{t}))=\mathtt{p}^2,\ \text{ if }\ \omega^*\in\mathbf{\Omega}^*,0\le l<k(\omega^*),\\ \omega^*(l\beta+\mathtt{t})\in\overline{\Lambda}\ \text{ and }\ \widetilde{t}^{(1)}(\omega^*(l\beta+\mathtt{t});\mathtt{t})=\mathtt{p}^2.\end{aligned}$$

And so on: this procedure is iterated until we exhaust all points in $\{\mathbf{\Omega}^*\}(\mathtt{t})\cap\overline{\Lambda}$. (Recall, their number and their positions vary with $\mathtt{t}\in[0,\beta]$.) At the end, we obtain a resulting function $\widetilde{t}=\widetilde{t}_{\mathrm{L}}(\,\cdot\,;\mathtt{t})$:

$$y\in\mathbb{R}^2\mapsto\widetilde{t}(y)\ \text{ where }\ \widetilde{t}(y)=\widetilde{t}^{(0)}(y)\wedge\widetilde{m}(y)\tag{5.20}$$

where

$$\widetilde{m}(y)=\widetilde{m}_{\mathrm{L}}(y;\mathtt{t})=\bigwedge_{l:\,\omega^*(l\beta+\mathtt{t})\in\{\mathbf{\Omega}^*\}(\mathtt{t})}m_{\omega^*(l\beta+\mathtt{t}),\mathtt{p}(\omega^*(l\beta+\mathtt{t}))\times\mathtt{s}}(y).\tag{5.21}$$

Here, we set:

$$\mathtt{p}(\omega^*(l\beta+\mathtt{t}))=0 \ \text{ when } \ \omega^*(l\beta+\mathtt{t})\in\overline{\Lambda}^{\rm c}.$$

Observe that

$$\widetilde{t}_{\rm L}(y;\mathtt{t})=0 \text{ for } y\in\overline{\Lambda}^{\rm c} \ \text{ and } \ \widetilde{t}_{\rm L}(y;\mathtt{t})=1 \text{ for } y\in\Lambda_{\rm R(L)}. \tag{5.22}$$

The Jacobian $J^{\pm}_{\rm L}(\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}})$ of the transform $\mathtt{T}^{\pm}_{\rm L}$ turns out to be of the form:

$$\begin{aligned} J^{\pm}_{\rm L}(\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}}) = \exp\Bigg[\int_0^\beta \mathtt{dt} \sum_{\omega^*\in\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}}} \\ \times \sum_{0\le l<k^*(\omega)} \ln\left(1\pm\mathtt{s}\times\big(\partial^1\widetilde{t}_{\rm L}\big)(\omega^*(l\beta+\mathtt{t});\mathtt{t})\right)\Bigg] \end{aligned} \tag{5.23}$$

where $(\partial^1\widetilde{t}_{\rm L})(y)$ stands for the partial derivative $\dfrac{\partial\widetilde{t}_{\rm L}}{\partial\mathtt{y}^1}(y;\mathtt{t})$, $y=(\mathtt{y}^1,\mathtt{y}^2)$. (The fact that the functions $\widetilde{t}_{\rm L}$ are non-differentiable on sets of positive co-dimension is not an obstacle here because of involvement of Wiener's integration.) The crucial quantity $\Big[J^+_{\rm L}(\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}})J^-_{\rm L}(\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}})\Big]^{1/2}$ in Equation (4.11) becomes

$$\begin{aligned} &\Big[J^+_{\rm L}(\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}})J^-_{\rm L}(\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}})\Big]^{1/2}= \\ &\exp\left(\int_0^\beta \mathtt{dt}\sum_{\omega^*\in\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}}}\sum_{0\le l<k^*(\omega)}\ln\left\{1-\Big[\mathtt{s}^2\big(\partial^1\widetilde{t}_{\rm L}\big)(\omega^*(l\beta+\mathtt{t});\mathtt{t})\Big]^2\right\}\right). \end{aligned} \tag{5.24}$$

We see that the quantity (5.24) is close to 1 when we are able to check that the sum

$$\sum_{\omega^*\in\Omega^*_\Lambda\vee\mathbf{\Omega}^*_{\Lambda^{\rm c}}}\sum_{0\le l<k^*(\omega)}\int_0^\beta \mathtt{dt}\Big[\big(\partial^1\widetilde{t}_{\rm L}\big)(\omega^*(l\beta+\mathtt{t});\mathtt{t})\Big]^2 \tag{5.25}$$

is close to 0.

We conclude this section with a straightforward assertion justifying the definition (5.3) that introduces the intermediate square $\overline{\Lambda}$.

Lemma 4. *Consider the events*

$$\mathcal{L}^{(1)}_{\rm L}=\{\mathbf{\Omega}^*\in\mathcal{W}^*_a(\mathbb{R}^2):\ \alpha_{\mathbb{R}^2\setminus\overline{\Lambda}}(\omega^*)=1\ \ \forall\,\omega^*\in\mathbf{\Omega}^*\ \ \textit{with}\ \ x(\omega^*)\in\Lambda^{\rm c}\} \tag{5.25a}$$

and

$$\mathcal{L}^{(2)}_{\rm L}=\{\mathbf{\Omega}^*\in\mathcal{W}^*_a(\mathbb{R}^2):\ \alpha_{\Lambda_{\rm R(L)}}(\omega^*)=1\ \ \forall\,\omega^*\in\mathbf{\Omega}^*\ \ \textit{with}\ \ x(\omega^*)\in\Lambda_0\} \tag{5.25b}$$

In other words, (a) for $\mathbf{\Omega}^*\in\mathcal{L}^{(1)}_{\rm L}$, every loop ω^* from $\mathbf{\Omega}^*$ which starts at a point $x(\omega^*)$ outside square Λ does not reach square $\overline{\Lambda}$, while (b) for $\mathbf{\Omega}^*\in\mathcal{L}^{(2)}_{\rm L}$, every loop ω^*_0 from $\mathbf{\Omega}^*_{\Lambda_0}$ (which starts in Λ_0) does not leave square $\Lambda_{\rm R(L)}$. Then, under condition (1.1.14),

$$\lim_{{\rm L}\to\infty}\mu_{\rm L}(\mathcal{L}^{(1)}_{\rm L})=\lim_{{\rm L}\to\infty}\mu_{\rm L}(\mathcal{L}^{(2)}_{\rm L})=1, \tag{5.26}$$

$\forall\,\mu\in\mathfrak{K}(\mathtt{z},\beta)$.

Proof. Both relations are proved in a similar way, so we discuss in detail one of them, say $\lim\limits_{L\to\infty} \mu_L(\mathcal{L}_L^{(1)}) = 1$. At first, we write

$$\mu(\mathcal{W}_a^* \setminus \mathcal{L}_L^{(1)}) = \mu(\exists \text{ at least one loop } \omega^* \text{ with } x(\omega^*) \in \Lambda^c \text{ reaching } \overline{\Lambda})$$
$$\leq \int \mu(\mathrm{d}\boldsymbol{\Omega}^*) \sum_{\omega^* \in \boldsymbol{\Omega}^*_{\Lambda^c}} \mathbf{1}\Big(\omega^*(\mathtt{t}) \in \overline{\Lambda} \text{ for some } \mathtt{t} \in [0, k(\omega^*)\beta]\Big).$$

By virtue of the Campbell theorem, the last integral equals

$$\int \mathrm{d}\omega^* \mathrm{M}(\omega^*) \mathbf{1}\Big(x(\omega^*) \in \Lambda^c \text{ but } \omega^*(\mathtt{t}) \in \overline{\Lambda} \text{ for some } \mathtt{t} \in [0, k(\omega^*)\beta]\Big)$$

which by the Ruelle superstability bound (2.3.19) does not exceed

$$\int_{\Lambda^c} \mathrm{d}x \int_{\mathcal{W}^*(x)} \mathbb{P}_x(\mathrm{d}\omega^*) \frac{\rho^{k(\omega^*)}}{k(\omega^*)} \mathbf{1}\Big(\omega^*(\mathtt{t}) \in \overline{\Lambda} \text{ for some } \mathtt{t} \in [0, k(\omega^*)\beta]\Big),$$

with $\rho := \mathrm{z}e^{\beta \mathrm{W}^-}$; cf. (1.1.14).

Next, we observe that the loop ω^* with the endpoint $x = (\mathrm{x}^1, \mathrm{x}^2) \in \Lambda^c$ (i.e., with $\max |\mathrm{x}^j|_{\mathrm{m}} \geq L$) can reach $\overline{\Lambda}$ only if at least one of its one-dimensional components (i.e., a scalar Brownian bridge with the endpoint x^j, $j = 1$ or 2) deviates from its origin by at least $(|\mathrm{x}^j| - \mathrm{L}) + \mathrm{L}^{3/4}$. Therefore, the last displayed expression is upper-bounded by

$$\begin{aligned} &2 \times 2 \sum_{k \geq 1} \frac{\rho^k}{k\sqrt{2\pi\beta k}} \int_{\mathrm{L}^{3/4}}^{\infty} \mathrm{d}\mathrm{x} \exp\left[-4\mathrm{x}^2/(2k\beta)\right] \\ &\quad \leq \sum_{k \geq 1} \frac{4\rho^k}{\sqrt{2\pi}\, k} \frac{\exp\left[-\mathrm{L}^{3/2}/(2k\beta)\right]}{L^{3/4}/\sqrt{k\beta} + \sqrt{\mathrm{L}^{3/2}/(k\beta) + 4/\pi}}. \end{aligned} \tag{5.27}$$

Here, we have used an estimate for the (scalar) Brownian bridge $\mathtt{B}(\mathtt{t})$ with endpoints $\mathtt{0}$ and $\mathtt{y} \geq 0$: $\forall\, \mathtt{A} > \mathtt{y}$

$$\overline{\mathrm{P}}^{\beta k}_{0,\mathrm{y}} \Big\{ \sup\, [\mathtt{B}(\mathtt{t}) :\ 0 \leq \mathtt{t} \leq \beta k] \geq \mathtt{A} \Big\} = \frac{1}{\sqrt{2\pi\beta k}} e^{-(2\mathtt{A}-y)^2/(2\beta k)} \tag{5.28a}$$

plus bounds for the tail of the normal distribution (see [30], Formula (3)):

$$\frac{e^{-\mathtt{A}^2/2}}{\mathtt{A} + \sqrt{\mathtt{A}^2 + 2}} \leq \int_{\mathtt{A}}^{\infty} e^{-\mathtt{t}^2/2} \mathrm{d}\mathtt{t} \leq \frac{\mathrm{e}^{-\mathtt{A}^2/2}}{\mathtt{A} + \sqrt{\mathtt{A}^2 + 4/\mathfrak{B}}}. \tag{5.28b}$$

It is not hard to see that the RHS of (5.27) tends to 0 as $\mathrm{L} \to \infty$. This completes the proof. □

In what follows, we will assume that a loop configuration $\boldsymbol{\Omega}^*$ lies in $\mathcal{L}_{\mathrm{L}}$. Together with (5.22), this will imply that the loops $\omega^* \in \boldsymbol{\Omega}^*$ with $x(\omega^*) \in \Lambda^c$ remains unaffected by transformations $\mathtt{T}^{\pm}(s)$.

6. Estimates for the Jacobians

To guarantee properties (I) and (IIIa) of Theorem 9, we need to secure that the good set $\mathcal{G}_{\mathrm{L}}$ carries a large measure and contains only those loop configurations $\boldsymbol{\Omega}^* \in \mathcal{W}^*(\mathbb{R}^2)$ for which the expression

$J_{\rm L}^+(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c})J_{\rm L}^-(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c})$ can be appropriately controlled. To this end, consider a random variable $\Sigma^{\rm J}(\mathbf{\Omega}^*) = \Sigma^{\rm J}_{\rm L}(\mathbf{\Omega}^*)$ given by the RHS of (5.25):

$$\begin{aligned}\Sigma^{\rm J}(\mathbf{\Omega}^*) : \mathbf{\Omega}^* \mapsto & \int_0^\beta {\rm d}\mathtt{t} \sum_{x\in\{\mathbf{\Omega}^*\}(\mathtt{t})} \Big[(\partial^1\widetilde{t}_{\rm L})(x;\mathtt{t})\Big]^2 \\ &= \sum_{\omega^*\in\mathbf{\Omega}^*} \int_0^\beta {\rm d}\mathtt{t} \sum_{0\leq l<k^*(\omega)} \Big[(\partial^1\widetilde{t}_{\rm L})(\omega^*(l\beta+\mathtt{t});\mathtt{t})\Big]^2.\end{aligned} \tag{6.1}$$

The formal definition of the set $\mathcal{G}_{\rm L}$ will require that the quantity $\Sigma^{\rm J}(\mathbf{\Omega}^*)$ is small (more precisely that some majorants for $\Sigma^{\rm J}(\mathbf{\Omega}^*)$ are small); see below. Formally, the property that $J_{\rm L}^+(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c})J_{\rm L}^-(\Omega^*_\Lambda \vee \mathbf{\Omega}^*_{\Lambda^c})$ is close to 1 follows from

Lemma 5. *If $\epsilon > 0$ is chosen small enough then the mean-value of $\Sigma^{\rm J}(\mathbf{\Omega}^*)$ vanishes as ${\rm L} \to \infty$:*

$$\lim_{{\rm L}\to\infty} \int \mu({\rm d}\mathbf{\Omega}^*)\Sigma^{\rm J}(\mathbf{\Omega}^*) = 0. \tag{6.2}$$

Proof. Let us start with technical definitions. Given $\mathtt{t} \in [0,\beta]$ and $x, x' \in \{\mathbf{\Omega}^*\}(t)$, we write:

$$\left.\begin{array}{ll} x \leftrightarrow x' & \text{whenever } a < |x-x'| < a+\epsilon \\ \text{and} & \\ x \xleftrightarrow{\{\mathbf{\Omega}^*\}(\mathtt{t})} x'' & \text{when there exists a collection of particles} \\ & x_0,\ldots,x_m \in \mathbf{\Omega}^* \text{ such that point } x_0 \text{ coincides} \\ & \text{with } x, \text{ point } x_m \text{ with } x'' \text{ and } \forall\, i=1,\ldots,m \\ & x_{i-1} \text{ and } x_i \text{ satisfy } a < |x_i - x_{i-1}| < a+\epsilon. \end{array}\right\} \tag{6.3}$$

Recall, the values $\mathtt{z}, \beta > 0$ are such that the bound (1.1.14) is satisfied. Referring below to a small $\epsilon > 0$, we mean conditions like this:

$$\frac{1}{4}\pi[(a+2\epsilon)^2 - a^2](1\vee\beta)\left(1 \vee \sum_{k\geq 1}\rho^k k\Big/(2\pi\beta)\right) < 1. \tag{6.4}$$

Constants $C_j \in (0,\infty)$ appearing in the argument vary with β and $\mathtt{z}$ (through ρ) but are independent of L.

To assess the integral in (6.2), observe that one possibility for value $\widetilde{t}_{\rm L}(\omega^*(l\beta+\mathtt{t}))$ is $\widetilde{t}_{\rm L}^{(0)}(\omega^*(l\beta+\mathtt{t}))$; the opposite case is where $\widetilde{t}_{\rm L}(\omega^*(l\beta+\mathtt{t}))$ equals $\widetilde{m}_{\rm L}(\omega^*(l\beta+\mathtt{t}))$. See Equations (5.8) and (5.20). In the former case, we have to deal with the derivative

$$\partial^1\widetilde{t}_{\rm L}^{(0)}(\omega^*(l\beta+\mathtt{t});\mathtt{t}) = T_{\rm L}(|\omega^*(l\beta+\mathtt{t})|_{\rm m})$$

where

$$T_{\rm L}(r) = \frac{[q\,(r - {\rm R(L)} - \epsilon - a/2)]^2}{\big[Q(\overline{\rm L} - {\rm R(L)} - \epsilon - a/2)\big]^2}\,\mathbf{1}(0 \leq r \leq \overline{\rm L}). \tag{6.5}$$

In the second case, we obtain that

$$\widetilde{t}_{\rm L}(\omega^*(l\beta+\mathtt{t})) = \widetilde{m}_{\rm L}(\omega^*(l\beta+\mathtt{t})),$$

and we have to use the structure of function $\widetilde{m}_{\rm L}(\omega^*(l\beta+\mathtt{t}))$ (related to multiple minimisation as defined in Equation (3.21)) to assess its derivative (cf. Section 6.7 in [12]).

All in all, to verify (6.2) it suffices to check that

$$\lim_{\mathrm{L}\to\infty}\int \mu(\mathrm{d}\mathbf{\Omega}^*)[\Sigma^{(1)}(\mathbf{\Omega}^*)+\Sigma^{(2)}(\mathbf{\Omega}^*)]=0. \tag{6.6}$$

Here, variable $\Sigma^{(1)}=\Sigma^{(1)}_{\mathrm{L}}$ is given by

$$\Sigma^{(1)}(\mathbf{\Omega}^*) \;=\int_0^\beta \mathtt{dt}\sum_{x\in\{\mathbf{\Omega}^*\}(\mathtt{t})} J_{\mathrm{L}}(|x|_{\mathrm{m}})=\sum_{\omega^*\in\mathbf{\Omega}^*}\int_0^{k(\omega^*)\beta} J_{\mathrm{L}}(|\omega^*(\mathtt{t})|_{\mathrm{m}})\mathtt{dt} \tag{6.7}$$

and corresponds to the first of the aforementioned possibilities (where we have $\widetilde{t}_{\mathrm{L}}(\omega^*(l\beta+\mathtt{t}))=\widetilde{t}^{(0)}_{\mathrm{L}}(\omega^*(l\beta+\mathtt{t}))$). Next, variable $\Sigma^{(2)}=\Sigma^{(2)}_{\mathrm{L}}$ corresponds to the second possibility and has the form

$$\begin{aligned}\Sigma^{(2)}(\mathbf{\Omega}^*)=\int_0^\beta \mathtt{dt}&\sum_{\substack{x,x',x''\in\{\mathbf{\Omega}^*\}(\mathtt{t})\\ x\neq x'}}\mathbf{1}\Big(x\leftrightarrow x'\Big)\mathbf{1}\Big(x\overset{\{\mathbf{\Omega}^*\}(\mathtt{t})}{\longleftrightarrow}x''\Big)\mathbf{1}\Big(|x|_{\mathrm{m}}\leq|x''|_{\mathrm{m}}\Big)\\ &\times\Big[\tau_{\mathrm{L}}(|x|_{\mathrm{m}}-\epsilon-a/2)-\tau_{\mathrm{L}}(|x''|_{\mathrm{m}})\Big]^2:=\Sigma^{(2,1)}(\mathbf{\Omega}^*)+\Sigma^{(2,2)}(\mathbf{\Omega}^*).\end{aligned} \tag{6.8}$$

The composition of the RHS is related to a 'cluster' structure accompanying the multiple minimization procedure in (5.21) which determines the value of interest $\widetilde{m}_{\mathrm{L}}(\omega^*(l\beta+\mathtt{t}))$. Formally, as follows from the definition, behind the indicator $\mathbf{1}\Big(x\overset{\{\mathbf{\Omega}^*\}(\mathtt{t})}{\longleftrightarrow}x''\Big)$ there is a 'chain' of points from the $\mathtt{t}$-section $\{\mathbf{\Omega}^*\}(\mathtt{t})$ which joins the 'extreme' points x and x'' (cf. Section 8 in [12] (whose notation system is partially followed here)).

Moreover, the partition $\Sigma^{(2)}(\mathbf{\Omega}^*)=\Sigma^{(2,1)}(\mathbf{\Omega}^*)+\Sigma^{(2,2)}(\mathbf{\Omega}^*)$ reflects the fact that x and x'', the two extreme points in the chain, can belong to the same loop ω^* or to two distinct loops, ω^* and $\omega^{*\prime\prime}$. More precisely, the summand $\Sigma^{(2,1)}=\Sigma^{(2,1)}_{\mathrm{L}}$ is specified as the sum

$$\begin{aligned}\sum_{\omega^*\in\mathbf{\Omega}^*}\int_0^\beta \mathtt{dt}&\sum_{0\leq l,l''<k(\omega^*)}\mathbf{1}\left(\omega^*(l\beta+\mathtt{t})\overset{\{\mathbf{\Omega}^*\}(\mathtt{t})}{\longleftrightarrow}\omega^*(l''\beta+\mathtt{t})\right)\\ &\times\Big[\mathbf{1}(\omega^*(l\beta+\mathtt{t})\leftrightarrow\omega^*(l''\beta+\mathtt{t}))\mathbf{1}(l\neq l'')\\ &\quad+\sum_{\omega^{*\prime}\neq\omega^{*\prime\prime}}\sum_{0\leq l'<k(\omega^{*\prime})}\mathbf{1}(\omega^*(l\beta+\mathtt{t})\leftrightarrow\omega^{*\prime}(l'\beta+\mathtt{t}))\Big]\\ &\quad\times\mathbf{1}\Big(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}}\leq|\omega^*(l''\beta+\mathtt{t})|_{\mathrm{m}}\Big)\\ &\times\Big[\tau_{\mathrm{L}}(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}}-\epsilon-a/2)-\tau_{\mathrm{L}}(|\omega^*(l''\beta+\mathtt{t})|_{\mathrm{m}})\Big]^2\end{aligned} \tag{6.9a}$$

whereas the term $\Sigma^{(2,2)} = \Sigma_{\rm L}^{(2,2)}$ equals the sum

$$\begin{aligned}
&\sum_{\omega^*,\omega^{*\prime\prime}\in\mathbf{\Omega}^*}\int_0^\beta \mathtt{d}\mathtt{t} \sum_{\substack{0\le l<k(\omega^*)\\ 0\le l''<k(\omega^{*\prime\prime})}} \mathbf{1}\left(\omega^*(l\beta+\mathtt{t})\overset{\{\mathbf{\Omega}^*\}(\mathtt{t})}{\longleftrightarrow}\omega^{*\prime\prime}(l''\beta+\mathtt{t})\right)\\
&\qquad\times\Big[\mathbf{1}(\omega^*(l\beta+\mathtt{t})\leftrightarrow\omega^{*\prime\prime}(l''\beta+\mathtt{t}))\\
&\qquad+\sum_{0\le l'<k(\omega^{*\prime\prime})}\mathbf{1}(l'\neq l'')\mathbf{1}(\omega^*(l\beta+\mathtt{t})\leftrightarrow\omega^{*\prime\prime}(l'\beta+\mathtt{t}))\\
&\qquad+\sum_{\omega^{*\prime}\neq\omega^{*\prime\prime}}\sum_{0\le l'<k(\omega^{*\prime})}\mathbf{1}(\omega^*(l\beta+\mathtt{t})\leftrightarrow\omega^{*\prime}(l'\beta+\mathtt{t}))\Big]\\
&\qquad\times\mathbf{1}\Big(|\omega^*(l\beta+\mathtt{t})|_{\rm m}\le|\omega^{*\prime\prime}(l''\beta+\mathtt{t})|_{\rm m}\Big)\\
&\qquad\times\Big[\tau_{\rm L}(|\omega^*(l\beta+\mathtt{t})|_{\rm m}-\epsilon-a/2)-\tau_{\rm L}(|\omega^{*\prime\prime}(l''\beta+\mathtt{t})|_{\rm m})\Big]^2.
\end{aligned}\tag{6.9b}$$

□

Proposition 1. *The mean value of* $\Sigma^{(1)}$ *is assessed as follows:*

$$\int_{\mathcal{W}_a^*(\mathbb{R}^2)}\mu(\mathrm{d}\mathbf{\Omega}^*)\Sigma_{\rm L}^{(1)}(\mathbf{\Omega}^*)\le C_0\Gamma(\mathrm{L})\tag{6.9c}$$

where $C_0\in(0,\infty)$ *is a constant and the quantity* $\Gamma(\mathrm{L})$ *is defined as follows:*

$$\Gamma(\mathrm{L}):=\int_\Lambda\frac{q(|x|_{\rm m}-\mathrm{R(L)}-\epsilon-a/2)^2}{Q(\mathrm{L}-\mathrm{R(L)}-\epsilon-a/2)^2}\mathrm{d}x,\ \ \textit{with}\ \lim_{\mathrm{L}\to\infty}\Gamma(\mathrm{L})=0.\tag{6.10}$$

Proof. To explain the bound (6.10), we first write, by the Campbell theorem:

$$\int_{\mathcal{W}_a^*(\mathbb{R}^2)}\mu(\mathrm{d}\mathbf{\Omega}^*)\Sigma_{\rm L}^{(2)}(\mathbf{\Omega}^*)=\int_{\mathcal{W}_a^{*,\rm f}(\mathbb{R}^2)}\mathrm{d}\omega^*\mathrm{M}(\omega^*)\int_0^{k(\omega^*)\beta}J_{\rm L}(|\omega^*(\mathtt{t})|_{\rm m})\mathtt{d}\mathtt{t}.\tag{6.11}$$

By the Ruelle superstability bound (2.3.19), the RHS does not exceed

$$\int_{\mathcal{W}_a^{*,\rm f}(\mathbb{R}^2)}\mathrm{d}\omega^*\frac{\rho^{k(\omega^*)}}{k(\omega^*)}\int_0^{k(\omega^*)\beta}J_{\rm L}(|\omega^*(\mathtt{t})|_{\rm m})\mathtt{d}\mathtt{t}.\tag{6.12}$$

When $x(\omega^*)\in\Lambda_{\mathrm{R(L)}}$, we estimate

$$T_{\rm L}(|\omega^*(\mathtt{t})|_{\rm m})\le\frac{1}{[Q(\mathrm{L}-\mathrm{R(L)}-\epsilon-a/2)]^2};\tag{6.13}$$

consequently, the corresponding contribution

$$\int_{\mathcal{W}_a^{*,\rm f}(\mathbb{R}^2)}\mathrm{d}\omega^*\mathbf{1}(x(\omega^*)\in\Lambda_{\mathrm{R(L)}})\frac{\rho^{k(\omega^*)}}{k(\omega^*)}\int_0^{k(\omega^*)\beta}J_{\rm L}(|\omega^*(\mathtt{t})|_{\rm m})\mathtt{d}\mathtt{t}$$

does not exceed

$$\begin{aligned}
&\frac{(2\mathrm{R(L)})^2}{[Q(\mathrm{L}-\mathrm{R(L)})-\epsilon-a/2)]^2}\sum_{k\ge1}\frac{(k\beta)\rho^k}{(2\pi k\beta)k}=\frac{\big(\log\,\log\,L\big)^{3/2}}{2\pi\,[Q(\mathrm{L}-\mathrm{R(L)}-\epsilon-a/2)]^2}\sum_{k\ge1}\frac{\rho^k}{k}\\
&\quad<\frac{\rho\big(\log\,\log\,L\big)^{3/2}}{2\pi(1-\rho)\,[Q(\mathrm{L}-\mathrm{R(L)}-\epsilon-a/2)]^2}.
\end{aligned}\tag{6.14}$$

This idea can be pushed further: we use estimate (6.14) whenever loop ω^* reaches $\Lambda_{\mathrm{R(L)}}$. For given $x \notin \Lambda_{\mathrm{R(L)}}$ and $\omega^* \in \mathcal{W}^*(x)$, this can occur when either (i) $k(\omega^*)$ is large—say, $k(\omega^*) > \big[|x|_{\mathrm{m}} - \mathrm{R(L)}\big]/2$—or when (ii) the opposite inequality $k(\omega^*) \leq \big[|x|_{\mathrm{m}} - \mathrm{R(L)}\big]/2$ holds true but the loop ω^* deviates from x, in the max-distance, by at least $|x|_{\mathrm{m}} - \mathrm{R(L)}$. Then the corresponding part of expression (6.14)

$$\begin{aligned}\int_{\mathcal{W}^*} &\mathrm{d}\omega^* \mathbf{1}(x(\omega^*) \notin \Lambda_{\mathrm{R(L)}})\\ &\times \mathbf{1}(\omega^*(\mathtt{t}) \in \Lambda_{\mathrm{R(L)}} \text{ for some } t \in [0, k(\omega^*)\beta])\\ &\qquad\times \frac{\rho^{k(\omega^*)}}{k(\omega^*)} \int_0^{k(\omega^*)\beta} J_{\mathrm{L}}(|\omega^*(\mathtt{t})|_{\mathrm{m}})\mathrm{d}\mathtt{t}\end{aligned}$$

is upper-bounded by

$$\begin{aligned}\int_{\mathbb{R}^2} \mathrm{d}x \Big\{ &\sum_{k \geq |x|_{\mathrm{m}}/2} \frac{(k\beta)\rho^k}{(2\pi k\beta)k} + \sum_{1 \leq k \leq |x|_{\mathrm{m}}/2} \frac{(k\beta)\rho^k}{k} \int_{\mathcal{W}^{k\beta}(0)} \mathbb{P}_0^{k\beta}(\mathrm{d}\omega^*)\\ &\times \mathbf{1}\Big(\max\, \big[|\omega^*(\mathtt{t})|_{\mathrm{m}} :\ 0 \leq \mathtt{t} \leq k\beta\big] > |x|_{\mathrm{m}} \Big) \Big\}.\end{aligned} \tag{6.15}$$

The first sum in (6.15) is evaluated through a convergent geometric progression:

$$\sum_{k \geq |x|_{\mathrm{m}}/2} \frac{\rho^k}{2\pi k} \leq \frac{\rho^{|x|_{\mathrm{m}}/2}}{2\pi(1-\overline{\rho})},$$

and its contribution into the integral $\int_{\mathbb{R}^2} \mathrm{d}x$ does not exceed a constant. To estimate the second sum, one can use the inequalities (5.28a,b). This yields:

$$\begin{aligned}\sum_{1 \leq k \leq |x|_{\mathrm{m}}^{1/2}} &\frac{(k\beta)\rho^k}{k} \int_{\mathcal{W}^{k\beta}(0)} \mathbb{P}_0^{k\beta}(\mathrm{d}\omega^*) \mathbf{1}\Big(\max\, \big[|\omega^*(\mathtt{t})|_{\mathrm{m}} :\ 0 \leq \mathtt{t} \leq k\beta\big] > |x|_{\mathrm{m}} \Big)\\ &\leq \frac{2}{|x|_{\mathrm{m}} + \sqrt{|x|_{\mathrm{m}}^2 + 4/\pi}} \frac{e^{-|x|_{\mathrm{m}}/\beta}}{2\pi(1-\rho)}.\end{aligned} \tag{6.16}$$

Consequently, the contribution of this sum to $\int_{\mathbb{R}^2} \mathrm{d}x$ also does not exceed a constant.

More generally, for a given $r > \mathrm{R(L)}$ we consider the contribution into (6.14) from loops ω^* with $x(\omega^*) \notin \Lambda_r$ such that $|\omega^*(\mathtt{t})|_{\mathrm{m}} = r$ for some $\mathtt{t} \in [0, k(\omega^*)\beta]$. Repeating the above argument, we conclude that this contribution again is less than or equal to a constant times $J_{\mathrm{L}}(r)$. Note that all constants can be made uniform; this implies that

$$\begin{aligned}(6.14) \leq &\frac{C_0}{[Q(\mathrm{L} - \mathrm{R(L)} - \epsilon - a/2)]^2}\\ &\times \Big[\big(\log\log\, L\big)^{3/2} + \int_{\mathrm{R(L)}}^{L} [q\,(r - \mathrm{R(L)} - \epsilon - a/2)]^2\, \mathrm{d}r \Big].\end{aligned} \tag{6.17}$$

As in [12], the quantity in the RHS of (6.17) (which is $= C_0 \times \Gamma(\mathrm{L})$) goes to 0 as $\mathrm{L} \to \infty$. This finishes the proof. □

It is instructive to note that the relation (6.9) does not require a smallness for ϵ.

We now pass to random variable $\Sigma_{\mathrm{L}}^{(2)} = \Sigma_{\mathrm{L}}^{(2,1)} + \Sigma_{\mathrm{L}}^{(2,2)}$.

Proposition 2. *For small enough ϵ,*

$$\lim_{\mathrm{L} \to \infty} \int_{\mathcal{W}_a^*(\mathbb{R}^2)} \mu(\mathrm{d}\mathbf{\Omega}^*) \Sigma_{\mathrm{L}}^{(2)}(\mathbf{\Omega}^*) = 0. \tag{6.18}$$

Proof. In the beginning, we again use the Campbell theorem (in conjunction with an argument similar to Equation (6.25) from [12]). Then the integral in (6.18) is less than or equal to a constant (say, C_4) times the sum $I^{2,1}+I^{2,2}$. Here the term $I^{2,1}=I^{2,1}_{\mathrm{L}}$ is specified as follows:

$$\begin{aligned}
I^{2,1} = &\int_0^\beta \mathrm{d}\mathtt{t} \int \mathrm{d}\omega^* \sum_{0\le l,l''<k(\omega^*)} \Big\{ \mathbf{1}(\omega^*(l\beta+\mathtt{t}) \leftrightarrow \omega_1^*(l''\beta+\mathtt{t}))\mathrm{M}(\omega^*) \\
&+ \sum_{m\ge 1}\prod_{1<i\le m} \int \mathrm{d}\omega_i^* \sum_{0\le l_i,\bar{l}_i<k(\omega_i^*)} \mathbf{1}(\omega_{i-1}^*(\bar{l}_{i-1}\beta+\mathtt{t}) \leftrightarrow \omega_i^*(l_i\beta+\mathtt{t})) \\
&\times \mathbf{1}(\omega_m^*(l_m\beta+\mathtt{t}) \leftrightarrow \omega^*(l''\beta+\mathtt{t}))\mathrm{M}(\omega^*,\omega_1^*,\ldots,\omega_m^*)\Big\} \\
&\quad \times \mathbf{1}\Big(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}} \le |\omega^*(l''\beta+\mathtt{t})|_{\mathrm{m}}\Big) \\
&\quad \times \Big[\tau_{\mathrm{L}}(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}} - \epsilon - a/2) - \tau_{\mathrm{L}}(|\omega^*(l''\beta+\mathtt{t})|_{\mathrm{m}})\Big]^2
\end{aligned} \tag{6.19a}$$

where the loop ω_0^* has been identified as ω^* and value $\bar{l}_0$ as l.

Likewise,

$$\begin{aligned}
I^{2,2} = &\int_0^\beta \mathrm{d}\mathtt{t} \int \mathrm{d}\omega^* \int \mathrm{d}\omega^{*\prime\prime} \\
&\times \sum_{\substack{0\le l<k(\omega^*)\\ 0\le l''<k(\omega^{*\prime\prime})}} \Big\{ \mathbf{1}(\omega^*(l\beta+\mathtt{t}) \leftrightarrow \omega^{*\prime\prime}(l''\beta+\mathtt{t}))\mathrm{M}(\omega^*,\omega^{*\prime\prime}) \\
&+ \sum_{m\ge 1}\prod_{1<i\le m} \int \mathrm{d}\omega_i^* \sum_{0\le l_i,\bar{l}_i<k(\omega_i^*)} \mathbf{1}(\omega_{i-1}^*(\bar{l}_{i-1}\beta+\mathtt{t}) \leftrightarrow \omega_i^*(l_i\beta+\mathtt{t})) \\
&\times \mathbf{1}(\omega_m^*(l_m\beta+\mathtt{t}) \leftrightarrow \omega^{*\prime\prime}(l''\beta+\mathtt{t}))\mathrm{M}(\omega^*,\omega_1^*,\ldots,\omega_m^*,\omega^{*\prime\prime})\Big\} \\
&\quad \times \mathbf{1}\Big(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}} \le |\omega^{*\prime\prime}(l''\beta+\mathtt{t})|_{\mathrm{m}}\Big) \\
&\quad \times \Big[\tau_{\mathrm{L}}(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}} - \epsilon - a/2) - \tau_{\mathrm{L}}(|\omega^{*\prime\prime}(l''\beta+\mathtt{t})|_{\mathrm{m}})\Big]^2
\end{aligned} \tag{6.19b}$$

where again the loop ω_0^* has been identified as ω^* and value $\bar{l}_0$ as l.

So, it suffices to verify that

$$\lim_{\mathrm{L}\to\infty} I^{2,1} = \lim_{\mathrm{L}\to\infty} I^{2,2} = 0.$$

Both integrals are analysed in a similar fashion, and we focus on one of them, say, $I^{2,2}$.

We use elementary bounds

$$\begin{aligned}
&\Big[\tau_{\mathrm{L}}(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}} - \epsilon - a/2) - \tau_{\mathrm{L}}(|\omega^{*\prime\prime}_m(l''\beta+\mathtt{t})|_{\mathrm{m}})\Big]^2 \\
&\quad \le \Big[|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}} - \epsilon - a/2 - |\omega^{*\prime\prime}(l''\beta+\mathtt{t}))|_{\mathrm{m}}\Big]^2 T_{\mathrm{L}}(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}})
\end{aligned} \tag{6.20a}$$

with T_{L} given in (6.5), and

$$\begin{aligned}
&\Big[|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}} - \epsilon - a/2 - |\omega^{*\prime\prime}(l''\beta+\mathtt{t}))|_{\mathrm{m}}\Big]^2 \\
&\quad \le 3(\epsilon+a/2)^2 + 3|\omega^*(l\beta+\mathtt{t})|^2_{\mathrm{m}} + 3|\omega^{*\prime\prime}(l''\beta+\mathtt{t}))|^2_{\mathrm{m}}.
\end{aligned} \tag{6.20b}$$

Employing in addition the Ruelle superstability bound (2.3.19), we conclude that (6.19b) does not exceed

$$\begin{aligned}
&3\int_0^\beta \mathtt{dt}\int \mathrm{d}\omega^*\frac{\overline{\rho}^{k(\omega^*)}}{k(\omega^*)}\int \mathrm{d}\omega^{*\prime\prime}\frac{\rho^{k(\omega^{*\prime\prime})}}{k(\omega^{*\prime\prime})}\\
&\times \sum_{\substack{0\le l<k(\omega^*)\\ 0\le l''<k(\omega^{*\prime\prime})}} T_{\mathrm{L}}(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}})\Big\{\mathbf{1}(\omega^*(l\beta+\mathtt{t})\leftrightarrow\omega^{*\prime\prime}(l''\beta+\mathtt{t}))\\
&\quad+\sum_{m\ge1}\prod_{1<i\le m}\int \mathrm{d}\omega_i^*\frac{\rho^{k(\omega_i^*)}}{k(\omega_i^*)}\sum_{0\le l_i,\bar{l}_i<k(\omega_i^*)}\mathbf{1}(\omega_{i-1}^*(\bar{l}_{i-1}\beta+\mathtt{t})\leftrightarrow\omega_i^*(l_i\beta+\mathtt{t}))\\
&\quad\times\mathbf{1}(\omega_m^*(l_m\beta+\mathtt{t})\leftrightarrow\omega^{*\prime\prime}(l''\beta+\mathtt{t}))\Big\}\mathbf{1}\Big(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}}\le|\omega^{*\prime\prime}(l''\beta+\mathtt{t})|_{\mathrm{m}}\Big)\\
&\qquad\times\Big[(\epsilon+a/2)^2+|\omega_0^*(l_0\beta+\mathtt{t})|_{\mathrm{m}}^2+|\omega_m^*(l_m\beta+\mathtt{t}))|_{\mathrm{m}}^2\Big].
\end{aligned}\tag{6.21}$$

Expanding the sum of squares in the parentheses, we obtain three expressions; in view of similarity of the argument used for analysing each of them, we focus on the one with the term $|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}}^2$:

$$\begin{aligned}
&\int_0^\beta \mathtt{dt}\int \mathrm{d}\omega^*\frac{\rho^{k(\omega^*)}}{k(\omega^*)}T_{\mathrm{L}}(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}})\int \mathrm{d}\omega^{*\prime\prime}\frac{\rho^{k(\omega^{*\prime\prime})}}{k(\omega^{*\prime\prime})}\\
&\quad\times\mathbf{1}\Big(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}}\le|\omega^{*\prime\prime}(l''\beta+\mathtt{t})|_{\mathrm{m}}\Big)\\
&\quad\times\sum_{\substack{0\le l<k(\omega^*)\\ 0\le l''<k(\omega^{*\prime\prime})}}|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}}^2\Big\{\mathbf{1}(\omega^*(l\beta+\mathtt{t})\leftrightarrow\omega^{*\prime\prime}(l''\beta+\mathtt{t}))\\
&\quad+\sum_{m\ge1}\prod_{1<i\le m}\int \mathrm{d}\omega_i^*\frac{\rho^{k(\omega_i^*)}}{k(\omega_i^*)}\sum_{0\le l_i,\bar{l}_i<k(\omega_i^*)}\mathbf{1}(\omega_{i-1}^*(\bar{l}_{i-1}\beta+\mathtt{t})\leftrightarrow\omega_i^*(l_i\beta+\mathtt{t}))\\
&\qquad\times\mathbf{1}(\omega_m^*(l_m\beta+\mathtt{t})\leftrightarrow\omega^{*\prime\prime}(l''\beta+\mathtt{t}))\Big\}.
\end{aligned}\tag{6.22}$$

Again, we can expand the curled brackets and will analyse the behavior of the most involved sum:

$$\begin{aligned}
&\int_0^\beta \mathtt{dt}\int \mathrm{d}\omega^*\frac{\rho^{k(\omega^*)}}{k(\omega^*)}\int \mathrm{d}\omega^{*\prime\prime}\frac{\rho^{k(\omega^{*\prime\prime})}}{k(\omega^{*\prime\prime})}\sum_{\substack{0\le l<k(\omega^*)\\ 0\le l''<k(\omega^{*\prime\prime})}}|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}}^2T_{\mathrm{L}}(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}})\\
&\quad\times\sum_{m\ge1}\prod_{1<i\le m}\int \mathrm{d}\omega_i^*\frac{\rho^{k(\omega_i^*)}}{k(\omega_i^*)}\sum_{0\le l_i,\bar{l}_i<k(\omega_i^*)}\mathbf{1}(\omega_{i-1}^*(\bar{l}_{i-1}\beta+\mathtt{t})\leftrightarrow\omega_i^*(l_i\beta+\mathtt{t}))\\
&\qquad\times\mathbf{1}(\omega_m^*(l_m\beta+\mathtt{t})\leftrightarrow\omega^{*\prime\prime}(l''\beta+\mathtt{t})).
\end{aligned}\tag{6.23}$$

The argument for estimating (6.23) starts with the analysis of the integral $\int \mathrm{d}\omega^{*\prime\prime}\frac{\rho^{k(\omega^{*\prime\prime})}}{k(\omega^{*\prime\prime})}$ for fixed values of the variables in the remaining integrals. To this end, we invoke the Fubini theorem and properties of the Brownian bridge. This allows us to conclude that

$$\begin{aligned}
&\int \mathrm{d}\omega^{*\prime\prime}\frac{\rho^{k(\omega^{*\prime\prime})}}{k(\omega^{*\prime\prime})}\sum_{0\le l''<k(\omega^{*\prime\prime})}\mathbf{1}(\omega_m^*(l_m\beta+\mathtt{t})\leftrightarrow\omega^{*\prime\prime}(l''\beta+\mathtt{t}))\le\\
&\int_{\mathbb{R}^2}\mathrm{d}x''\int_0^\beta \mathtt{dt}\int_{A[\omega_{m-1}(\mathtt{t}),\epsilon]}\mathrm{d}y\sum_{k''\ge1}\rho^{k''}\frac{e^{-|y-x''|^2/(2\mathtt{t})}}{2\pi}\frac{e^{-|y-x''|^2/(2(k''\beta-\mathtt{t}))}}{2\pi(k''\beta-\mathtt{t})}.
\end{aligned}\tag{6.24}$$

Here

$$A[\omega_{m-1}(\mathtt{t}),\epsilon]=\{y\in\mathbb{R}^2: a<|y-\omega_{m-1}(\mathtt{t})|<a+2\epsilon\}\tag{6.25}$$

stands for an annulus of width 2ϵ around the center $\omega_{m-1}(\mathtt{t})$. (Initially, point $y \in A[\omega_{m-1}(\mathtt{t}),\epsilon]$ emerges here as the point on the circle of radius $|y-\omega_{m-1}(\mathtt{t})|$ about $\omega_{m-1}(\mathtt{t})$ where the loop ω hits this circle while t is the hitting time.) The RHS of (6.24) yields a quantity $\leq C_3\epsilon$.

This argument can be iterated for the integrals $\int \mathrm{d}\omega_i^* \dfrac{\rho^{k(\omega_i^{*\prime\prime})}}{k(\omega_i^{*\prime\prime})}$, where we have to take into account the double sum $\sum\limits_{0\leq l_i,\bar{l}_i<k(\omega_i^*)}$. However, it only affects the constant in front of ϵ.

At the end, assuming that $\epsilon>0$ is small enough we arrive at the following bound for (6.23):

$$\frac{C_1\epsilon}{1-C_2\epsilon}\int_0^\beta \int \mathrm{d}\omega^* \frac{\rho^{k(\omega^*)}}{k(\omega^*)} \sum_{0\leq l<k(\omega^*)} |\omega^*(l\beta+\mathtt{t})|^2_{\mathrm{m}} T_{\mathrm{L}}(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}})\mathrm{d}\mathtt{t}. \tag{6.26}$$

The integral (6.26) is analysed in the same manner as in Proposition 1 (cf. (6.9)) and tends to 0. (The presence of the sum $\sum\limits_{0\leq l<k(\omega^*)}$ in (6.26) does not affect the core of the argument.)

This completes the proof of Proposition 2 and Lemma 5. □

7. Estimates for the Change in the Energy

In this section, we assess the expression (4.8) and complete the proof of Theorem 10. The argument is based on the same idea as in Section 8.6 of [12] (again, we partially borrow the system of notation from there). In the course of the argument, we will produce a further (and final) specification of the set $\mathcal{G}_{\mathrm{L}} \subset \mathcal{W}_a^*(\mathbb{R}^2)$ of good loop configurations. Namely, given $\mathbf{\Omega}^* \in \mathcal{W}_a^*(\mathbb{R}^2)$, we set, as before,

$$\mathbf{\Omega}^*_\Lambda = \{\omega^* \in \mathbf{\Omega}^* :\ x(\boldsymbol{\omega}^*) \in \Lambda\}, \quad \mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}} = \{\omega^* \in \mathbf{\Omega}^* :\ x(\boldsymbol{\omega}^*) \in \Lambda^{\mathrm{c}}\}.$$

Then write

$$\begin{aligned} &h(\mathtt{T}^+_{\mathrm{L}}(s)\mathbf{\Omega}^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}}) + h(\mathtt{T}^-_{\mathrm{L}}(s)\mathbf{\Omega}^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}}) - 2h(\mathbf{\Omega}^*_\Lambda|\mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}}) \\ &\quad = \int_0^\beta \mathrm{d}\mathtt{t}\Big\{E[\{\mathtt{T}^+(s)\mathbf{\Omega}^*_\Lambda\}(\mathtt{t})|\{\mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}}\}(\mathtt{t})] \\ &\qquad + E[\{\mathtt{T}^-(s)\mathbf{\Omega}^*_\Lambda\}(\mathtt{t})|\{\mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}}\}(\mathtt{t})] - 2E[\{\mathbf{\Omega}^*_\Lambda\}(\mathtt{t})|\{\mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}}\}(\mathtt{t})]\Big\}. \end{aligned} \tag{7.1}$$

Here, $E\left[\{\mathtt{T}^\pm(s)\mathbf{\Omega}^*\}_\Lambda(\mathtt{t})|\{\mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}}\}(\mathtt{t})\right]$ is defined as the sum

$$\begin{aligned} &\frac{1}{2}\sum_{x,x'\in\{\mathbf{\Omega}^*_\Lambda\}(\mathtt{t})} V\left(|x\pm s\tilde{t}(x) - x' \mp s\tilde{t}(x')|\right) \\ &\qquad + \sum_{\substack{x\in\{\mathbf{\Omega}^*_\Lambda\}(\mathtt{t}) \\ x'\in\{\mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}}\}(\mathtt{t})}} V\left(|x\pm s\tilde{t}(x) - x' \mp s\tilde{t}(x')|\right) \end{aligned} \tag{7.2}$$

while $E[\{\mathbf{\Omega}^*\}(\mathtt{t})|\mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}}(\mathtt{t})]$ is obtained by omitting the terms containing s (cf. (2.3.11) and (2.3.12)). Recall, our aim is to guarantee that on the good set $\mathcal{G}_{\mathrm{L}}$, the absolute value of the variable $\Sigma^h_{\mathrm{L}}(\mathbf{\Omega}^*)$ is small. Two straightforward bounds turn out to be helpful:

$$\begin{aligned} &\Big|E[\{\mathtt{T}^+(s)\mathbf{\Omega}^*_\Lambda\}(\mathtt{t})|\{\mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}}\}(\mathtt{t})] + E[\{\mathtt{T}^-(s)\mathbf{\Omega}^*_\Lambda\}(\mathtt{t})|\{\mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}}\}(\mathtt{t})] \\ &\qquad -2E[\{\mathbf{\Omega}^*_\Lambda\}(\mathtt{t})|\{\mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}}\}(\mathtt{t})]\Big| \\ &\leq \overline{V}^{(2)}\mathrm{s}^2\left\{\frac{1}{2}\sum_{x,x'\in\{\mathbf{\Omega}^*_\Lambda\}(\mathtt{t})} + \sum_{\substack{x\in\{\mathbf{\Omega}^*_\Lambda\}(\mathtt{t}) \\ x'\in\{\mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}}\}(\mathtt{t})}}\right\}|\tilde{t}(x)-\tilde{t}(x')|^2\mathbf{1}(|x-x'|\leq \mathrm{R}_0) \end{aligned} \tag{7.3}$$

and

$$\begin{aligned}&\frac{1}{3}|\widetilde{t}(x)-\widetilde{t}(x')|^2\\&\quad\leq|\widetilde{t}(x)-\tau_L(|x|_{\rm m})|^2+|\tau_{\rm L}(|x|_{\rm m})-\tau_{\rm L}(|x'|_{\rm m})|^2+|\tau_{\rm L}(|x'|_{\rm m})-\widetilde{t}(x')|^2.\end{aligned}\tag{7.4}$$

Recall, $\overline{V}^{(2)}$ has been defined in (1.1.4b). Then (7.3), (7.4) yield that

$$\begin{aligned}&\big|h(\mathtt{T}_{\rm L}^{+}(s)\boldsymbol{\Omega}_\Lambda^*|\boldsymbol{\Omega}_{\Lambda^{\rm c}}^*)+h(\mathtt{T}_{\rm L}^{-}(s)\boldsymbol{\Omega}_\Lambda^*|\boldsymbol{\Omega}_{\Lambda^{\rm c}}^*)-2h(\boldsymbol{\Omega}_\Lambda^*|\boldsymbol{\Omega}_{\Lambda^{\rm c}}^*)\big|\\&\quad\leq 3\overline{V}^{(2)}\mathtt{s}^2\int_0^\beta\mathtt{dt}\left\{\frac{1}{2}\sum_{x,x'\in\{\boldsymbol{\Omega}_\Lambda^*\}(\mathtt{t})}+\sum_{\substack{x\in\{\boldsymbol{\Omega}_\Lambda^*\}(\mathtt{t})\\x'\in\{\boldsymbol{\Omega}_{\Lambda^{\rm c}}^*\}(\mathtt{t})}}\right\}\mathbf{1}(|x-x'|\leq{\rm R}_0)\\&\quad\times\Big[2|\widetilde{t}(x)-\tau_L(|x|_{\rm m})|^2+|\tau_L(|x|_{\rm m})-\tau_L(|x'|_{\rm m})|^2\Big]=:\Sigma_{\rm L}^{(3)}(\boldsymbol{\Omega}^*)+\Sigma_{\rm L}^{(4)}(\boldsymbol{\Omega}^*)\end{aligned}\tag{7.5}$$

where variables $\Sigma_{\rm L}^{(3)}$ and $\Sigma_{\rm L}^{(4)}$ emerge when we expand the sum of squares in the parentheses.

As above, we will try to make sure that the expected values of variables $\Sigma_{\rm L}^{(3)}$ and $\Sigma_{\rm L}^{(4)}$ vanish as ${\rm L}\to\infty$:

Lemma 6.

$$\lim_{{\rm L}\to\infty}\int\mu({\rm d}\boldsymbol{\Omega}^*)\Sigma_{\rm L}^{(3)}(\boldsymbol{\Omega}^*)=\lim_{{\rm L}\to\infty}\int\mu({\rm d}\boldsymbol{\Omega}^*)\Sigma_{\rm L}^{(4)}(\boldsymbol{\Omega}^*)=0.\tag{7.6}$$

Proof. As before, we focus on one of the relations in Equation (7.6), say, for $\Sigma_{\rm L}^{(4)}$. It is instructive to expand

$$\Sigma_{\rm L}^{(4)}(\boldsymbol{\Omega}^*)=\Sigma_{\rm L}^{(4,1)}(\boldsymbol{\Omega}^*)+\Sigma_{\rm L}^{(4,2)}(\boldsymbol{\Omega}^*)$$

where $\Sigma_{\rm L}^{(4,1)}(\boldsymbol{\Omega}^*)$ gives a single-loop contribution to $\Sigma_{\rm L}^{(4)}(\boldsymbol{\Omega}^*)$ whereas $\Sigma_{\rm L}^{(4,2)}(\boldsymbol{\Omega}^*)$ yields a contribution from pairs of loops:

$$\begin{aligned}\Sigma_{\rm L}^{(4,1)}(\boldsymbol{\Omega}^*)=&\sum_{\omega^*\in\boldsymbol{\Omega}_\Lambda^*}\int_0^\beta\mathtt{dt}\Bigg\{\sum_{0\leq l<\bar{l}<k(\omega^*)}\\&\times\Big[\tau_{\rm L}\big(\big|\omega^*(l\beta+\mathtt{t})\big|_{\rm m}\big)-\tau_{\rm L}\big(\big|\omega^*(\bar{l}\beta+\mathtt{t})\big|_{\rm m}\big)\Big]^2\\&\quad\times\mathbf{1}\Big(\big|\omega^*(l\beta+\mathtt{t})-|\omega^*(\bar{l}\beta+\mathtt{t}\big|<{\rm R}_0\Big)\\&+\sum_{\omega^{*\prime}\in\boldsymbol{\Omega}_{\Lambda^{\rm c}}^*}\sum_{\substack{0\leq l<k(\omega^*)\\0\leq l'<k(\omega^{*\prime})}}\Big[\tau_{\rm L}\big(\big|\omega^*(l\beta+\mathtt{t})\big|_{\rm m}\big)-\tau_{\rm L}\big(\big|\omega^{*\prime}(l'\beta+\mathtt{t})\big|_{\rm m}\big)\Big]^2\\&\quad\times\mathbf{1}\Big(\big|\omega^*(l\beta+\mathtt{t})-|\omega^{*\prime}(l'\beta+\mathtt{t}\big|<{\rm R}_0\Big)\Bigg\}\end{aligned}\tag{7.7}$$

and

$$\begin{aligned}\Sigma_{\rm L}^{(4,2)}(\boldsymbol{\Omega}^*)=&\frac{1}{2}\int_0^\beta\mathtt{dt}\sum_{\substack{\omega^*,\omega^{*\prime}\in\boldsymbol{\Omega}_\Lambda^*\\\omega^*\neq\omega^{*\prime}}}\sum_{\substack{0\leq l<k(\omega^*)\\0\leq l'<k(\omega^{*\prime})}}\\&\times\Big[\tau_{\rm L}\big(\big|\omega^*(l\beta+\mathtt{t})\big|_{\rm m}\big)-\tau_{\rm L}\big(\big|\omega^{*\prime}(l'\beta+\mathtt{t})\big|_{\rm m}\big)\Big]^2\\&\quad\times\mathbf{1}\Big(\big|\omega^*(l\beta+\mathtt{t})-|\omega^{*\prime}(l'\beta+\mathtt{t}\big|<{\rm R}_0\Big).\end{aligned}\tag{7.8}$$

(The factor $3\overline{V}^{(2)}\mathtt{s}^2$ carried from (7.5) has been discarded.)

Following Equation (6.22) from [12], we estimate: (a) for $\left|\omega^*(l\beta+\mathtt{t})\right|_{\mathrm{m}} \leq \left|\omega^{*\prime}(l'\beta+\mathtt{t})\right|_{\mathrm{m}}$

$$\begin{aligned}&\left[\tau_{\mathrm{L}}\left(\left|\omega^*(l\beta+\mathtt{t})\right|_{\mathrm{m}}\right)-\tau_{\mathrm{L}}\left(\left|\omega^{*\prime}(l'\beta+\mathtt{t})\right|_{\mathrm{m}}\right)\right]^2\\ &\leq\left[\left|\omega^*(l\beta+\mathtt{t})\right|_{\mathrm{m}}-\left|\omega^{*\prime}(l'\beta+\mathtt{t})\right|_{\mathrm{m}}-\epsilon-a/2\right]^2 T_{\mathrm{L}}(|\omega^*(l\beta+\mathtt{t})|_{\mathrm{m}})\end{aligned}$$

and (b) $\left|\omega^{*\prime}(l'\beta+\mathtt{t})\right|_{\mathrm{m}} \leq \left|\omega^*(l\beta+\mathtt{t})\right|_{\mathrm{m}}$

$$\begin{aligned}&\left[\tau_{\mathrm{L}}\left(\left|\omega^*(l\beta+\mathtt{t})\right|_{\mathrm{m}}\right)-\tau_{\mathrm{L}}\left(\left|\omega^{*\prime}(l'\beta+\mathtt{t})\right|_{\mathrm{m}}\right)\right]^2\\ &\leq\left[\left|\omega^{*\prime}(l'\beta+\mathtt{t})\right|_{\mathrm{m}}-\left|\omega^*(l\beta+\mathtt{t})\right|_{\mathrm{m}}-\epsilon-a/2\right]^2 T_{\mathrm{L}}(|\omega^{*\prime}(l\beta+\mathtt{t})|_{\mathrm{m}})\end{aligned}$$

where T_{L} has been defined in (6.5).

After substituting these estimates in (7.2), the relation $\int \mu(\mathrm{d}\mathbf{\Omega}^*)\Sigma^{(4)}_{\mathrm{L}}(\mathbf{\Omega}^*) \to 0$ is verified in the same way as in Proposition 1. □

Lemma 6 (and the comments on other terms emerging from the bound (7.5)), together with Lemmas 4 and 5, allows us to define the set $\mathcal{G}_{\mathrm{L}}$. Namely,

$$\mathcal{G}_{\mathrm{L}}=\left\{\mathbf{\Omega}^*\in\mathcal{L}_{\mathrm{L}}:\ \Sigma^{(i)}_{\mathrm{L}}(\mathbf{\Omega}^*)<c,\ 1\leq i\leq 4\right\} \tag{7.9}$$

where $c\in(0,\infty)$ is a chosen constant (viz., $c=1/2$). Applying the Chebyshev inequality guarantees

Lemma 7. *For all $\delta\in(0,1)$ and $c\in(0,\infty)$, there exists $L^*_1\in(0,\infty)$ such that for $L>L^*_1$ the probability $\mu(\mathcal{G}_{\mathrm{L}})\geq 1-\delta$.*

A formal summary of properties of transformations $\mathtt{T}^{\pm}(s)$ is given in the following Theorem:

Theorem 11. *Given $\mathbf{\Omega}^*\in\mathcal{G}_{\mathrm{L}}$, the transformations $\mathtt{T}^{\pm}_{\mathrm{L}}(s)\ :\ \mathbf{\Omega}^*\mapsto\widetilde{\mathbf{\Omega}}^*\in\mathcal{W}^*_a(\mathbb{R}^2)$ possess the following properties:*

(i) The maps $\mathtt{T}^{\pm}(s)$ are measurable and $1-1$.

(ii) $\widetilde{\mathbf{\Omega}}^*_{\Lambda^{\mathrm{c}}}=\mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}}$ and $\widetilde{\mathbf{\Omega}}^*_{\Lambda_0}=\mathtt{S}(s)\mathbf{\Omega}^*_{\Lambda_0}$. Moreover, there exists a $1-1$ correspondence between the loops $\widetilde{\omega}^*\in\widetilde{\mathbf{\Omega}}^*_{\Lambda}$ and $\omega^*\in\mathbf{\Omega}^*_{\Lambda}$ such that $\widetilde{\omega}^*$ is obtained as a deformation of ω^* via tuned shifts of $\mathtt{t}$-sections, in the manner described in Section 3. In particular, $k(\widetilde{\omega}^*)=k(\omega^*)$.

(iii) Equation (4.6) holds, where the expression $\left[J^+_{\mathrm{L}}(\mathbf{\Omega}^*_{\Lambda}\vee\mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}})J^-_{\mathrm{L}}(\mathbf{\Omega}^*_{\Lambda}\vee\mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}})\right]^{1/2}$ is close to 1 uniformly in $\mathbf{\Omega}^*$ for L large.

(iv) The quantity $\exp\left[h(\mathtt{T}^+_{\mathrm{L}}(s)\mathbf{\Omega}^*_{\Lambda}|\mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}})+h(\mathtt{T}^-_{\mathrm{L}}(s)\mathbf{\Omega}^*_{\Lambda}|\mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}})-2h(\mathbf{\Omega}^*_{\Lambda}|\mathbf{\Omega}^*_{\Lambda^{\mathrm{c}}})\right]$ in (4.8) is close to 1 uniformly in $\mathbf{\Omega}^*$ for L large.

The assertions of Theorems 8 and 10 then follow.

8. Concluding Remarks and Future Research

The series of publications involving the present authors, [23,28,29] and [24,31], have been motivated, on the one hand, by a spectacular success on Mermin–Wagner type theorems, [6], achieved in the past for a broad class of two-dimensional classical and quantum systems and, on the other hand, by a recognised progress in experimental quantum physics creating and working with thin materials like graphene. There has been increasing interest in graphene since its discovery. Much research has been done on this linear dispersion and, in particular, on the transport properties of graphene. This may be a topic of future research. We intend also to elaborate the similar technique for the Hubbard model, which is a highly oversimplified model for strongly interacting electrons in a solid, in line with [28]. The Hubbard model is a kind of minimum model which takes into account quantum mechanical motion of electrons in a solid, and nonlinear repulsive interaction between electrons.

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